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Dual superconformal symmetry, and the amplitude/Wilson loop connection

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Abstract: We show that tree level superstring theories on certain supersymmetric backgrounds admit a symmetry which we call "fermionic T-duality". This is a non-local redefinition of the fermionic worldsheet fields similar to the redefinition we perform on bosonic variables when we do an ordinary T-duality. This duality maps a supersymmetric background to another supersymmetric background with different RR fields and a different dilaton. We show that a certain combination of bosonic and fermionic T-dualities maps the full superstring theory on $AdS_5 \times S^5$ back to itself in such a way that gluon scattering amplitudes in the original theory map to something very close to Wilson loops in the dual theory. This duality maps the "dual superconformal symmetry" of the original theory to the ordinary superconformal symmetry of the dual model. This explains the dual superconformal invariance of planar scattering amplitudes of $N = 4$ super Yang Mills and also sheds some light on the connection between amplitudes and Wilson loops. In the appendix, we propose a simple prescription for open superstring MHV tree amplitudes in a flat background.

Keywords: Supersymmetry and Duality, AdS-CFT Correspondence.

Contents

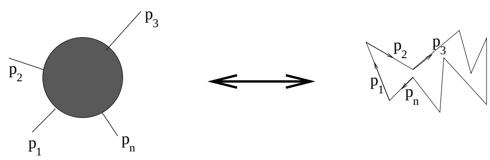

Figure 1: Relation between the amplitude and the Wilson loop. A planar scattering amplitude of n gluons is related to a Wilson loop computation involving an n sided polygonal Wilson loop where the sides are light like vectors given by the momenta.

1. Introduction

During the past year a surprising connection was found between planar scattering amplitudes and Wilson loops in $\mathcal{N} = 4$ super Yang Mills, for a recent review and a more complete set of references see [1] . This was first noticed in the strong coupling computation of the amplitudes in [2] . The connection that was found in [2] was apparently valid only at leading order in the strong coupling expansion. However, the same connection was soon found in weak coupling computations [3−6], based on previous amplitude computations in [7, 8]. More recently an impressive check of this relationship was performed at two loops for six gluons in [9, 10].

The basic statement of the relationship is as follows. One considers the color ordered amplitudes $\mathcal{A}(P_1, \cdots, p_n)$, defined via a color decomposition of the planar amplitude

$$\mathcal{A}(a_1, p_1, a_2, p_2, \cdots) = \sum_{\text{Permutations}} Tr[T^{a_1} \cdots T^{a_n}] \mathcal{A}(p_1, \cdots, p_n) \tag{1.1}$$

where a_i are the group indices and p_i are the momenta and we suppressed the polarization dependence. We can then write the MHV amplitudes as

$$\mathcal{A}_{\text{MHV}} = \mathcal{A}_{MHV, tree}\ \hat{\mathcal{A}}(p_1, \cdots, p_n) \tag{1.2}$$

where $\mathcal{A}_{MHV, tree}$ is the tree level MHV amplitude [11] . Then the observation is that

$$\hat{\mathcal{A}}(p_1, \cdots, p_n) = \langle W(p_1, \cdots, p_n) \rangle \tag{1.3}$$

where W is a Wilson loop that ends on a contour made by n lightlike segments, each proportional to p_i, see figure 1. To be more precise, the left hand side in (1.3) is infrared divergent and the right hand side is UV divergent. The structure of these divergencies is known. The statement is really about the finite parts of the amplitudes, which can have a complicated dependence on the kinematic invariants of the process.

A closely related fact is that scattering amplitudes display an interesting non-trivial symmetry called "dual conformal invariance". This symmetry was first found in perturbative computations in [12] , and it was recently also observed in next to MHV amplitudes in [13] , where it was promoted to a full "dual superconformal symmetry". One would also like to understand the origin of this symmetry. If one accepts the relationship between amplitudes and Wilson loops, then this symmetry is the ordinary superconformal symmetry acting on Wilson loops.

In this paper we show that one can understand this "dual superconformal symmetry" using a T-duality symmetry of the full superstring theory on $AdS_5 \times S^5$. The T-duality involves ordinary bosonic T-dualities, which were considered already in [2] , plus novel "fermionic" T-dualities. These fermionic T-dualities consist in certain non-local redefinitions of the fermionic variables of the superstring. The fermionic T-dualities change the dilaton and the RR fields without modifying the metric. After these T-dualities the sigma model looks the same as the original sigma model but the computation of the amplitude in the original model maps to a computation of an object very close to a Wilson loop in the dual theory. The ordinary superconformal invariance of the T-dual model is the "dual superconformal symmetry" of the original theory. The amplitude computation does not map precisely to a Wilson loop computation but to a certain computation involving D(-1) branes and strings stretching between them. For MHV amplitudes we expect, on the basis of perturbative computations [3, 4, 9, 10], that the difference should amount to a simple prefactor which is equal to the tree level MHV amplitude (1.2) . We will not derive this factor in this paper, but we will give some plausibility arguments.

We will discuss in some detail the nature of fermionic T-duality for general backgrounds. We will give some general rules regarding the transformation of the background fields under fermionic T-dualities. Fermionic T-dualities are possible when we have a supercharge that anticommutes to zero, $Q^2 = 0$. In that case one can represent the action of this supercharge as the shift of a certain fermionic coordinate of the sigma model $\theta \to \theta + \epsilon$. The fermionic T-duality is a transformation of the fermionic variables rather similar to the one we do for the case of bosonic T-dualities, in the sense that we redefine the field in such a way that we exchange the equation of motion with the Bianchi identity. The T-dual sigma model leads to a different background for the superstring. Thus a fermionic T-duality relates the superstring on a supersymmetric background to the superstring on another supersymmetric background. In general this fermionic T-duality is a valid symmetry only at string tree level for reasons similar to the ones that imply that a bosonic T-duality of a non-compact scalar is only a symmetry of certain tree level computations.

A connection between amplitudes and Wilson loops in momentum space was discussed in [14, 15]. One performs a Fourier transformation of an ordinary Wilson loop to obtain the Wilson loop in momentum space. Then the amplitude is related to a particular momentum space Wilson loop which looks exactly as the one in figure 1. The important point we are making here is that this *momentum* space Wilson loop can be computed by mapping it to an ordinary *position* space Wilson loop with the same shape.

It was also expected that "dual conformal symmetry" should be connected to integrability. In fact, in the simpler case of a bosonic AdS sigma model we show that the non-trivial generators in the "dual conformal group" correspond to some of the non-local charges that arise due to integrability. This conclusion has also been reached for the full $AdS_5 \times S^5$ theory in [16] .

In an appendix we also propose a simple prescription for computing MHV tree level open string scattering amplitudes in flat space. This prescription is related to the self-dual N=2 string [17, 18] and reproduces the amplitudes that have been computed previously in [19] using the standard formalism.

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This paper is organized as follows. In section two we introduce the concept of a "fermionic T-duality" and we explore some of its properties. In section three we perform a set of bosonic and fermionic transformations that map $AdS_5 \times S^5$ back to itself which maps the problem of amplitudes to a problem closely related to Wilson loops. In section four we discuss in more detail what the computation of the amplitude maps into. In section five we discuss the relation between the conformal symmetry of the dual theory and the non-local charges associated to integrability. In section six we present some conclusions.

In appendix A we discuss a proposal for computing MHV amplitudes in flat space string theory. This is disconnected from the rest of the paper and can be read on its own.

2. Fermionic T-duality

In this section, we discuss "fermionic T-duality" which is a generalization of the Buscher version of T-duality to theories with fermionic worldsheet scalars.[1] We first show how fermionic T-duality transforms background superfields in the Green-Schwarz and pure spinor sigma models. We then translate these transformations into the language of Type II supergravity fields and show that fermionic T-duality changes the background values of the dilaton and Ramond-Ramond fields. A simple example of fermionic T-duality relates the flat background with the self-dual graviphoton background. In the following section we will apply this transformation to the $AdS_5 \times S^5$ case.

2.1 Review of bosonic T-duality

In the sigma model description of T-duality, one starts with a sigma model

$$S = \int d^2z(g_{mn}(x) + b_{mn}(x))\partial x^m \bar{\partial} x^n \tag{2.1}$$

and assumes that the background fields g_{mn} and b_{mn} are invariant under the shift isometry

$$x^1 \to x^1 + c, \quad x^{\hat{m}} \to x^{\hat{m}} \tag{2.2}$$

where c is a constant and $\hat{m}$ ranges over all values except $m = 1$. Since x^1 only appears with derivatives, the action is

$$S = \int d^2z(g_{11}(\hat{x})\partial x^1\bar{\partial}x^1 + l_{1\hat{m}}(\hat{x})\partial x^1\bar{\partial}x^{\hat{m}} + l_{\hat{m}1}(\hat{x})\partial x^{\hat{m}}\bar{\partial}x^1 + l_{\hat{m}\hat{n}}(\hat{x})\partial x^{\hat{m}}\bar{\partial}x^{\hat{n}}) \tag{2.3}$$

where $l_{mn} = g_{mn} + b_{mn}$.

If g_{11} is nonzero, one can use the Buscher procedure [21] to T-dualize the sigma model with respect to x^1. This is done as follows. We first replace the derivatives of x^1 by a vector field $(A, \bar{A})$ on the worldsheet and we add a lagrange multiplier field $\tilde{x}^1$, that forces the vector field to be the derivative of a scalar

$$S = \int d^2z[g_{11}A\bar{A} + l_{1\hat{m}}A\bar{\partial}x^{\hat{m}} + l_{\hat{m}1}\partial x^{\hat{m}}\bar{A} + l_{\hat{m}\hat{n}}\partial x^{\hat{m}}\bar{\partial}x^{\hat{n}} + \tilde{x}^1(\partial\bar{A} - \bar{\partial}A)] \tag{2.4}$$

[1] T-dualities involving fermionic fields were considered in [20] , but in their case they were T-dualizing the phase of a fermionic field, which was essentially bosonic. Thus it is not obviously related to what we are doing here.

If we first integrate out the lagrange multiplier $\widetilde{x}^1$, we force $\partial\bar{A} - \bar{\partial}A = 0$ which can be solved by saying that $A = \partial x^1$ and $\bar{A} = \bar{\partial}x^1$ and we go back to the original model. On the other hand, if we first integrate out the vector field we obtain the T-dualized action

$$S = \int d^2z[g'_{11}\partial\widetilde{x}^1\bar{\partial}\widetilde{x}^1 + l'_{1\hat{m}}\partial\widetilde{x}^1\bar{\partial}x^{\hat{m}} + l'_{\hat{m}1}\partial x^{\hat{m}}\bar{\partial}\widetilde{x}^1 + l'_{\hat{m}\hat{n}}\partial x^{\hat{m}}\bar{\partial}x^{\hat{n}}] \tag{2.5}$$

where

$$\begin{aligned} &g'_{11} = (g_{11})^{-1}, \qquad && l'_{1\hat{m}} = (g_{11})^{-1}l_{1\hat{m}}, \qquad && l'_{\hat{m}1} = -(g_{11})^{-1}l_{\hat{m}1}, \\ &l'_{\hat{m}\hat{n}} = l_{\hat{m}\hat{n}} - (g_{11})^{-1}l_{\hat{m}1}l_{1\hat{n}}. \end{aligned} \tag{2.6}$$

Furthermore, the measure factor coming from integration over the bosonic vector field will induce a change in the dilaton ϕ by [21, 22]

$$\phi' = \phi - \frac{1}{2}\log g_{11}. \tag{2.7}$$

In the above discussion we have not said whether x^1 is compact or not. In order for the transformation to be valid on an arbitrary compact Riemman surface, it is important that x^1 is compact. The reason is that on an arbitrary surface, the condition that the field strength of the vector field is zero does not imply that it is the gradient of a scalar. The vector field could have holonomies on the various cycles of the Riemann surface. If the Lagrange multiplier field $\widetilde{x}^1$ is a compact field that can have winding on these circles, then we find that, after integrating it out, it imposes that the holonomy of the vector field has certain integral values. In this case we can still write the vector field in terms of a scalar x^1, which might wind along the cycles of the Riemman surface.

If we are considering the theory on the sphere or the disk, we do not need to worry about this and we can perform this transformation even for non-compact scalars, as long as the external vertex operators do not carry momentum. Note that in this case we can always write a vector field with zero field strength in terms of the gradient of a scalar. If we are on the sphere and the external vertex operators carry momentum, then the T-dual problem does not correspond to anything we ordinarily encounter in string theory. The situation is nicer in the case of the disk with external states that carry momentum only at the boundary of the disk. In this case, after the T-duality these open string states carry winding and we can interpret them as stretching between different D-branes that are localized in the T-dual coordinate. In general we will get as many D-branes as insertions we have on the boundary. In this case we need to treat the zero modes of the scalars separately. The original model contains an integration of the zero mode of the scalars which needs to be done before doing the T-duality. Correspondingly in the T-dual model we do not integrate over the zero mode of the T-dualized scalar, we just set it to some arbitrary value at some point on the boundary of the disk. This fixes the position of one of the D-branes on the T-dual circle. The other D-brane positions are fixed by the momenta that the vertex operators carried in the original theory.

In summary, even though a bosonic T-duality for a non-compact scalar is not well defined to all orders in string perturbation theory, one can do it for the disk diagram (and also for the sphere if none of the particles carries momentum in the original direction).

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2.2 Sigma model in superspace and fermionic T-duality

Suppose one is now given a Green-Schwarz-like sigma model depending on bosonic and fermionic worldsheet variables (x^m, θ^μ) such that the worldsheet action is invariant under a constant shift of one of the fermionic variables θ^1. In other words, the action is invariant under

$$\theta^1 \to \theta^1 + \rho, \quad x^m \to x^m, \quad \theta^{\tilde{\mu}} \to \theta^{\tilde{\mu}} \tag{2.8}$$

where ρ is a fermionic constant and $\tilde{\mu}$ ranges over all fermionic variables except for θ^1. Of course such backgrounds preserve a supersymmetry, whose properties we will discuss in more detail below.

Invariance under (2.8) implies that θ^1 only appears in the action with derivatives as $\partial\theta^1$ or $\bar{\partial}\theta^1$, so the worldsheet action has the form

$$S = \int d^2z[B_{11}(Y)\partial\theta^1\bar{\partial}\theta^1 + L_{1M}(Y)\partial\theta^1\bar{\partial}Y^M + L_{M1}(Y)\partial Y^M\bar{\partial}\theta^1 + L_{MN}\partial Y^M\bar{\partial}Y^N] \tag{2.9}$$

where $Y^M = (x^m, \theta^{\tilde{\mu}})$, $M = (m, \tilde{\mu})$ ranges over all indices except for $\mu = 1$, and $L_{MN}(Y) = G_{MN}(Y) + B_{MN}(Y)$ is the sum of the graded-symmetric tensor G_{MN} and the graded-antisymmetric tensor B_{MN}.

If $B_{11}(Y)$ is nonzero, one can use the Buscher procedure to T-dualize the sigma model with respect to θ^1. This is done by first introducing a fermionic vector field $(A, \bar{A})$. We replace the derivatives of θ^1 by the fermionic vector field. In addition we introduce the lagrange multiplier field $\widetilde{\theta}^1$ which imposes that the vector field is the derivative of a fermionic scalar via a term $\int d^2z\widetilde{\theta}^1(\partial\bar{A} - \bar{\partial}A)$. The resulting action is

$$\begin{aligned} S = \int d^2z[&B_{11}(Y)A\bar{A} + L_{1M}(Y)A\bar{\partial}Y^M + L_{M1}(Y)\partial Y^M\bar{A} + L_{MN}\partial Y^M\bar{\partial}Y^N \\ &+ \tilde{\theta}^1(\partial\bar{A} - \bar{\partial}A)] \end{aligned} \tag{2.10}$$

Integrating out $\tilde{\theta}^1$ imposes that $A = \partial\theta^1$ and $\bar{A} = \bar{\partial}\theta^1$. On the other hand, when we first integrate out the fermionic gauge field we obtain the T-dualized action

$$S = \int d^2z[B'_{11}(Y)\partial\widetilde{\theta}^1\bar{\partial}\widetilde{\theta}^1 + L'_{1M}(Y)\partial\widetilde{\theta}^1\bar{\partial}Y^M + L'_{M1}(Y)\partial Y^M\bar{\partial}\widetilde{\theta}^1 + L'_{MN}\partial Y^M\bar{\partial}Y^N] \tag{2.11}$$

where

$$\begin{aligned} B'_{11} &= -(B_{11})^{-1}, \qquad L'_{1M} = (B_{11})^{-1}L_{1M}, \qquad L'_{M1} = (B_{11})^{-1}L_{M1}, \\ L'_{MN} &= L_{MN} - \frac{1}{B_{11}}L_{1N}L_{M1} \end{aligned} \tag{2.12}$$

Furthermore, the measure factor coming from integration over the fermionic vector field will induce a change in the dilaton ϕ by

$$\phi' = \phi + \frac{1}{2}\log B_{11}. \tag{2.13}$$

since the integration of the vector field has exactly the same formal form as the one we had for the bosonic T-duality, except that in this case we are integrating over an anticommuting variable. Thus, the change in ϕ under fermionic T-duality has the opposite sign from the change in ϕ under bosonic T-duality. Another difference with bosonic T-duality is that fermionic T-duality does not change the relative sign of $\bar\partial\theta^1/\bar\partial\widetilde{\theta}^1$ versus $\partial\theta^1/\partial\widetilde{\theta}^1$, and does not change the relative sign of L'_{1M}/L_{1M} versus L'_{M1}/L_{M1}. We can find the explicit on-shell relation between the original and the T-dualized variables by computing the equations of motion for $(A,\bar A)$ and using the equation of motion for the field $\widetilde{\theta}^1$ which implies that the vector field is given by the gradient of θ^1. We find

$$\bar\partial\widetilde{\theta}^1 = B_{11}\bar\partial\theta^1 - (-1)^{s(M)}L_{1M}\bar\partial Y^M, \quad \partial\widetilde{\theta}^1 = B_{11}\partial\theta^1 + L_{M1}\partial Y^M, \tag{2.14}$$

where $s(M)=0$ if M is bosonic and $S(M)=1$ if M is fermionic. On the other hand the equations that relate a boson to the T-dual boson, coming from (2.4) , are

$$\bar\partial\widetilde{x}^1 = -(g_{11}\bar\partial x^1 + l_{1\hat m}\bar\partial x^{\hat m}), \quad \partial\widetilde{x}^1 = g_{11}\partial x^1 + l_{\hat m 1}\partial x^{\hat m}. \tag{2.15}$$

In other words, we have $d\widetilde{x}^1 = g_{11} * dx^1 + \cdots$ for the boson while we have $d\widetilde{\theta}^1 = Bd\theta^1 + \cdots$ for the fermion. Notice the absence of the $*$ for the fermionic case.

Note that the fermionic variables are morally non-compact. Our arguments here have ignored the fact that the vector field can have non-trivial holonomies on the Riemann surface. Thus our derivation is only justified in the case of the disk but not on higher genus Riemann surfaces. Even on the disk, we will need to treat the zero modes of the original and the T-dual fermion in a special way. We will integrate over the zero modes of the initial fermion before doing the T-duality and we will not integrate over the fermion zero modes of the T-dual fermions. (This is similar to the treatment of non-compact bosonic zero modes on the disk.)

If one wanted to define fermionic T-duality on a higher genus Riemann surface, one would need to introduce fermionic variables which are allowed to be non-periodic when its worldsheet location z is taken around a non-trivial cycle on the surface. Note that the usual Green-Schwarz θ variables are defined to be periodic and satisfy

$$\theta(z + C_i) = \theta(z) \tag{2.16}$$

where C_i is any non-trivial cycle on the worldsheet. If one wants to require that the fermionic vector field $(A,\bar A)$ has trivial holonomies so that it can be expressed as the gradient of θ, one would need to use a Lagrange multiplier term $\int d^2z\ \tilde\theta(\partial\bar A - \bar\partial A)$ where $\tilde\theta(z)$ is a non-periodic variable satisfying

$$\tilde\theta(z + C_i) = \tilde\theta(z) + \rho_i, \tag{2.17}$$

and ρ_i are Grassmann constants which need to be integrated over.

So if the original fermionic variable is periodic, the dual fermionic variable is non-periodic and contains an extra zero mode for every non-trivial cycle on the worldsheet. Similarly, if the original fermionic variable is non-periodic, the dual fermionic variable will be periodic and the holonomies of the vector field around the non-trivial cycles will correspond to the ρ_i constants in (2.17). This T-dual relation between periodic and non-periodic fermionic variables is analogous to the T-dual relation between non-compact bosonic variables and bosonic variables compactified on a circle of zero radius.

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2.3 T-duality in pure spinor formalism

Although one normally does not expect two-derivative terms for fermions such as $\int d^2zB_{11}\partial\theta^1\bar\partial\theta^1$, these terms arise in Green-Schwarz and pure spinor sigma models for Type II superstrings in Ramond-Ramond backgrounds. To find how the T-duality transformations of (2.12) act on the Type II supergravity background fields, one needs to know the relation of $L_{MN}(Y)$ with the onshell supergravity fields. In the Green-Schwarz formalism, this relation depends on the choice of superspace torsion constraints and can be quite complicated. As recently discussed in [23], the most convenient method for determining this relation is to use the pure spinor formalism where BRST invariance determines the choice of torsion constraints and allows a straightforward identification of the background fields.

In the pure spinor version of the Type II sigma model, the worldsheet action is

$$\frac{1}{2\pi\alpha'}\int d^2z[L_{MN}(Z)\partial Z^M\bar\partial Z^N+P^{\alpha\hat\beta}(Z)d_\alpha\hat d_{\hat\beta}+E^\alpha_M(Z)d_\alpha\bar\partial Z^M+E^{\hat\alpha}_M(Z)\partial Z^M\hat d_{\hat\alpha}$$
$$+\Omega^\beta_{M\alpha}(Z)\lambda^\alpha w_\beta\bar\partial Z^M+\hat\Omega^{\hat\beta}_{M\hat\alpha}(Z)\partial Z^M\hat\lambda^{\hat\alpha}\hat w_{\hat\beta}+C^{\beta\hat\gamma}_\alpha(Z)\lambda^\alpha w_\beta\hat d_{\hat\gamma}+\hat C^{\hat\beta\gamma}_{\hat\alpha}(Z)d_\gamma\hat\lambda^{\hat\alpha}\hat w_{\hat\beta}$$
$$+S^{\beta\hat\delta}_{\alpha\hat\gamma}(Z)\lambda^\alpha w_\beta\hat\lambda^{\hat\gamma}\hat w_{\hat\delta}+w_\alpha\bar\partial\lambda^\alpha+\hat w_{\hat\alpha}\partial\hat\lambda^{\hat\alpha}]+\frac{1}{4\pi}\int d^2z\Phi(Z)\mathcal{R} \qquad (2.18)$$

where Z^M are coordinates for N=2 d=10 superspace, d_α and $\hat d_{\hat\alpha}$ are independent fermionic variables, $(\lambda^\alpha,w_\alpha)$ and $(\hat\lambda^{\hat\alpha},\hat w_{\hat\alpha})$ are the left and right-moving pure spinor ghosts, and $\mathcal{R}$ is the worldsheet curvature. BRST invariance implies relations between the various superfields appearing in (2.18) where the BRST operators are $Q=\int dz\ \lambda^\alpha d_\alpha$ and $\hat Q=\int d\bar z\ \hat\lambda^{\hat\alpha}\hat d_{\hat\alpha}$. By comparing with the vertex operators for massless fields, one learns that the $\theta=\hat\theta=0$ component of $P^{\alpha\hat\beta}$ is $P^{\alpha\hat\beta}|_{\theta=\hat\theta=0}=-\frac{i}{4}e^\phi F^{\alpha\hat\beta}$ where $F^{\alpha\hat\beta}$ is the Ramond-Ramond field strength in bispinor notation,[2] the $\theta=\hat\theta=0$ components of E^α_m and $E^{\hat\alpha}_m$ are the N=2 d=10 gravitinos, the $\theta=\hat\theta=0$ components of $\Omega^\beta_{m\alpha}(\gamma^{ab})^\alpha_\beta\pm\hat\Omega^{\hat\beta}_{m\hat\alpha}(\gamma^{ab})^{\hat\alpha}_{\hat\beta}$ are the spin connection and NS-NS three-form, the $\theta=\hat\theta=0$ components of $C^{\beta\hat\gamma}_\alpha(\gamma^{ab})^\alpha_\beta$ and $\hat C^{\hat\beta\gamma}_{\hat\alpha}(\gamma^{ab})^{\hat\alpha}_{\hat\beta}$ are the N=2 gravitino field-strengths, and the $\theta=\hat\theta=0$ component of $S^{\beta\hat\delta}_{\alpha\hat\gamma}(\gamma^{ab})^\alpha_\beta(\gamma^{cd})^{\hat\gamma}_{\hat\delta}$ is the Riemann tensor and the derivative of the NS-NS three-form.

If (2.18) is invariant under the fermionic shift in (2.8), one can easily apply the Buscher procedure of the previous subsection to the action of (2.18). One finds that (2.18) is T-dualized to

$$\frac{1}{2\pi\alpha'}\int d^2z[B'_{11}(Y)\partial\tilde\theta^1\bar\partial\tilde\theta^1+L'_{1M}(Y)\partial\tilde\theta^1\bar\partial Y^M+L'_{M1}(Y)\partial Y^M\bar\partial\tilde\theta^1+L'_{MN}\partial Y^M\bar\partial Y^N+$$
$$+P'^{\alpha\hat\beta}(Y)d_\alpha\hat d_{\hat\beta}+E'^\alpha_1(Y)d_\alpha\bar\partial\tilde\theta^1+E'^\alpha_M(Y)d_\alpha\bar\partial Y^M+E'^{\hat\alpha}_1(Y)\partial\tilde\theta^1\hat d_{\hat\alpha}+E'^{\hat\alpha}_M(Y)\partial Y^M\hat d_{\hat\alpha}+\ldots]$$
$$+\frac{1}{4\pi}\int d^2z\Phi'(Y)\mathcal{R} \qquad (2.19)$$

[2]The relation to the usual notation for the RR field strengths of type IIB string theory is $F^{\alpha\hat\beta}=(\gamma^m)^{\alpha\hat\beta}F_m+\frac{1}{3!}(\gamma^{m_1m_2m_3})^{\alpha\hat\beta}F_{m_1m_2m_3}+\frac{1}{2}\frac{1}{5!}(\gamma^{m_1\cdots m_5})^{\alpha\hat\beta}F_{m_1\cdots m_5}$. The factor of e^ϕ in $P=-\frac{i}{4}e^\phi F$ is present since P has the kinetic term $\int d^{10}xe^{-2\phi}P^2$.

where Y^M ranges over all bosonic and fermionic variables except for θ^1, the superfields $[B'_{11}, L'_{1M}, L'_{M1}, L'_{MN}, \Phi']$ are defined as in (2.12), and

$$
\begin{aligned}
P'^{\alpha\hat\beta} &= P^{\alpha\hat\beta} - (B_{11})^{-1} E_1^\alpha E_1^{\hat\beta}, & E_1'^\alpha &= (B_{11})^{-1} E_1^\alpha, & E_1'^{\hat\alpha} &= (B_{11})^{-1} E_1^{\hat\alpha}, \\
E_M'^\alpha &= E_M^\alpha - (B_{11})^{-1} L_{1M} E_1^\alpha, & E_M'^{\hat\alpha} &= E_M^{\hat\alpha} - (B_{11})^{-1} E_1^{\hat\alpha} L_{M1}, \\
\Omega'^{\beta}_{1\alpha} &= (B_{11})^{-1} \Omega^\beta_{1\alpha}, & \hat\Omega'^{\hat\beta}_{1\hat\alpha} &= (B_{11})^{-1} \hat\Omega^{\hat\beta}_{1\hat\alpha}, \\
\Omega'^{\beta}_{M\alpha} &= \Omega^\beta_{M\alpha} - (B_{11})^{-1} L_{1M} \Omega^\beta_{1\alpha}, & \hat\Omega'^{\hat\beta}_{M\hat\alpha} &= \hat\Omega^{\hat\beta}_{M\hat\alpha} - (B_{11})^{-1} \hat\Omega^{\hat\beta}_{1\hat\alpha} L_{M1}, \\
C'^{\beta\hat\gamma}_\alpha &= C^{\beta\hat\gamma}_\alpha - (B_{11})^{-1} E_1^{\hat\gamma} \Omega^\beta_{1\alpha}, & \hat C'^{\hat\beta\gamma}_{\hat\alpha} &= \hat C^{\hat\beta\gamma}_{\hat\alpha} - (B_{11})^{-1} \hat\Omega^{\hat\beta}_{1\hat\alpha} E_1^\gamma, \\
S'^{\beta\hat\delta}_{\alpha\hat\gamma} &= S^{\beta\hat\delta}_{\alpha\hat\gamma} - \hat\Omega^{\hat\delta}_{1\hat\gamma} \Omega^\beta_{1\alpha}.
\end{aligned}
\tag{2.20}
$$

Note that the worldsheet variables in the BRST operators $Q = \int dz \lambda^\alpha d_\alpha$ and $\hat Q = \int d\bar z \hat\lambda^{\hat\alpha} \hat d_{\hat\alpha}$ are not affected by fermionic T-duality, so BRST invariance is manifestly preserved. Although the fermionic T-duality transformations of (2.20) are similar to the bosonic T-duality transformations discussed in [23], there are some crucial differences. For example, $E_1'^\alpha$ has the same relative sign as $E_1'^{\hat\alpha}$ in(2.20). But in bosonic T-duality if one dualizes the x^p coordinate as in [23],

$$
E_p'^\alpha = (G_{pp})^{-1} E_p^\alpha, \quad E_p'^{\hat\alpha} = -(G_{pp})^{-1} E_p^{\hat\alpha}. \tag{2.21}
$$

As will now be explained, this difference implies that unlike bosonic T-duality, fermionic T-duality does not exchange the Type IIA and Type IIB superstrings and does not modify the dimension of the D-brane.

As discussed in [24], the pure spinor Type II sigma model and BRST operators are invariant under three independent local Lorentz transformations which transform

$$
\begin{aligned}
\delta E_M^a &= L^a_b E_M^b, & \delta E_M^\alpha &= M^{ab} (\gamma_{ab})^\alpha_\beta E_M^\beta, & \delta E_M^{\hat\alpha} &= \hat M^{ab} (\gamma_{ab})^{\hat\alpha}_{\hat\beta} E_M^{\hat\beta}, \\
\delta d_\alpha &= M^{ab} (\gamma_{ab})_\alpha^\beta d_\beta, & \delta \hat d_{\hat\alpha} &= \hat M^{ab} (\gamma_{ab})_{\hat\alpha}^{\hat\beta} \hat d_{\hat\beta}, & \ldots
\end{aligned}
\tag{2.22}
$$

where M^{ab} and $\hat M^{ab}$ are independent of L^{ab} and ... denotes similar transformations on all background fields and worldsheet fields with tangent-space spinor indices. Furthermore, it was shown in [24] that BRST invariance of the sigma model implies the superspace torsion constraints

$$
T^a_{\alpha\beta} = i\ f^a_b\ \gamma^b_{\alpha\beta}, \quad T^a_{\hat\alpha\hat\beta} = i\ \hat f^a_b\ \gamma^b_{\hat\alpha\hat\beta}, \tag{2.23}
$$

where f^a_b and $\hat f^a_b$ are $O(9,1)$ matrices.

To compare with the usual description of Type II supergravity which has the torsion constraints

$$
T^a_{\alpha\beta} = i\ \gamma^a_{\alpha\beta}, \quad T^a_{\hat\alpha\hat\beta} = i\ \gamma^a_{\hat\alpha\hat\beta}, \tag{2.24}
$$

one can use the local Lorentz symmetries of M^{ab} and $\hat M^{ab}$ to gauge-fix f^a_b and $\hat f^a_b$. After gauge-fixing, only the combined Lorentz symmetry of all three types of indices together is preserved, which is the usual local Lorentz symmetry of supergravity. If f^a_b and $\hat f^a_b$ are

SO(9, 1) matrices with determinant +1, one can gauge $f_b^a = \hat{f}_b^a = \delta_b^a$ and recover (2.24). But to recover (2.24) when f_b^a (or $\hat{f}_b^a$) has determinant -1, one needs to flip the chirality of the unhatted (or hatted) spinor.

After performing bosonic T-duality (say in a flat background) with respect to the coordinates $(x^1, \ldots, x^p)$, the relative minus sign in the transformation of E_M^α versus $E_M^{\hat{\alpha}}$ in (2.21) implies that the components $(f_1^1, \ldots, f_p^p)$ of f_b^a have opposite sign with respect to the components $(\hat{f}_1^1, \ldots, \hat{f}_p^p)$ of $\hat{f}_b^a$. So to return to the standard torsion constraints of (2.24), one needs to perform local Lorentz transformations using M^{ab} and $\hat{M}^{ab}$ which cancel this change in relative sign in f versus $\hat{f}$. These local Lorentz transformations modify in the expected manner the D-brane boundary conditions which relate hatted and unhatted spinors. Furthermore, if p is odd, the determinants of f and $\hat{f}$ will have opposite sign. So to recover the torsion constraints of (2.24), one will have to flip the chirality of either the hatted or unhatted spinors, which switches the Type IIA and Type IIB superstring.

On the other hand, since in fermionic T-duality there are no relative minus signs in the transformation of E_M^α versus $E_M^{\hat{\alpha}}$, one does not need to perform local Lorentz rotations to return to the constraints of (2.24). So there is no switch of Type IIA and Type IIB superstrings, and no modification of the dimension of the D-brane.

2.4 Transformations of component fields

By considering the $\theta = \hat{\theta} = 0$ components of the superfields in (2.20), one finds that the fermionic T-duality transformations leave invariant the NS-NS fields g_{mn} and b_{mn}, and transform the Ramond-Ramond bispinor field-strength $F^{\alpha\hat{\beta}}$ and dilaton ϕ as

$$-\frac{i}{4} e^{\phi'} F'^{\alpha\hat{\beta}} = -\frac{i}{4} e^{\phi} F^{\alpha\hat{\beta}} - \epsilon^\alpha \hat{\epsilon}^{\hat{\beta}} C^{-1}, \quad \phi' = \phi + \frac{1}{2} \log C \tag{2.25}$$

where C is the $\theta = \hat{\theta} = 0$ component of B_{11} and $(\epsilon^\alpha, \hat{\epsilon}^{\hat{\alpha}})$ are the $\theta = \hat{\theta} = 0$ components of $(E_1^\alpha, E_1^{\hat{\alpha}})$. Although it is not difficult to also work out the T-duality transformations of the fermionic fields, we will assume here that all fermionic background fields have been set to zero.

To determine the relation of C and $(\epsilon^\alpha, \epsilon^{\hat{\alpha}})$ with the supergravity fields, note that the torsion constraints imply that the superspace 3-form field-strength

$$H_{ABC} = E_A^M E_B^N E_C^P H_{MNP} = E_A^M E_B^N E_C^P \partial_{[M} B_{NP]} \tag{2.26}$$

has constant spinor-spinor-vector components [25]

$$H_{\alpha\beta c} = i(\gamma_c)_{\alpha\beta}, \quad H_{\hat{\alpha}\hat{\beta}c} = -i(\gamma_c)_{\hat{\alpha}\hat{\beta}}, \quad H_{\alpha\hat{\beta}c} = 0, \tag{2.27}$$

where $A = (c, \alpha, \hat{\alpha})$ denotes tangent-superspace indices, M denotes curved-superspace indices, and E_A^M is the inverse super-vierbein. (The relative minus sign in $H_{\alpha\beta c}$ versus $H_{\hat{\alpha}\hat{\beta}c}$ is because $H \to -H$ under a worldsheet parity transformation which switches $z \to \bar{z}$ and $\alpha \to \hat{\alpha}$.)

Since the fermionic isometry implies that $\partial_1 B_{1m} = 0$ where $\partial_1 \equiv \frac{\partial}{\partial\theta^1}$, one finds that

$$\begin{aligned} \partial_m C &= \partial_m B_{11}|_{\theta=\hat{\theta}=0} = H_{11m}|_{\theta=\hat{\theta}=0} = E_1^A E_1^B E_m^C H_{ABC}|_{\theta=\hat{\theta}=0} \\ &= i\epsilon^\alpha \epsilon^\beta e_m^c (\gamma_c)_{\alpha\beta} - i\hat{\epsilon}^{\hat{\alpha}} \hat{\epsilon}^{\hat{\beta}} e_m^c (\gamma_c)_{\hat{\alpha}\hat{\beta}} = i\epsilon\gamma_m\epsilon - i\hat{\epsilon}\gamma_m\hat{\epsilon} \end{aligned} \tag{2.28}$$

where $e_m^c \equiv E_m^c|_{\theta=\hat{\theta}=0}$ is the usual vierbein, $E_1^\alpha|_{\theta=\hat{\theta}=0} = \epsilon^\alpha$ and $E_1^{\hat{\alpha}}|_{\theta=\hat{\theta}=0} = \hat{\epsilon}^{\hat{\alpha}}$.

Under the fermionic isometry of (2.8),

$$E_M^\alpha \delta Z^M = E_1^\alpha \rho, \qquad E_M^{\hat{\alpha}} \delta Z^M = E_1^{\hat{\alpha}} \rho \tag{2.29}$$

where ρ is a constant anticommuting parameter. Since the $\theta = \hat{\theta} = 0$ components of $E_M^\alpha \delta Z^M$ and $E_M^{\hat{\alpha}} \delta Z^M$ are the local supersymmetry parameters [26], and since $E_M^\alpha \delta Z^M|_{\theta=\hat{\theta}=0} = \epsilon^\alpha \rho$ and $E_M^{\hat{\alpha}} \delta Z^M|_{\theta=\hat{\theta}=0} = \hat{\epsilon}^{\hat{\alpha}} \rho$, the isometry of (2.8) implies that the component background supergravity fields are invariant under the supersymmetry transformation parameterized by the Killing spinors $\epsilon^\alpha = E_1^\alpha|_{\theta=\hat{\theta}=0}$ and $\hat{\epsilon}^{\hat{\alpha}} = E_1^{\hat{\alpha}}|_{\theta=\hat{\theta}=0}$. Note that we are talking about one supersymmetry given by these two spinors, and not two independent supersymmetries. So (2.28) implies that the derivative of C is related to the Killing spinors ϵ^α and $\hat{\epsilon}^{\hat{\alpha}}$. Note that the constant part of C is unconstrained, as can be seen from the fact that $B_{11} \partial \theta^1 \bar{\partial} \theta^1$ changes by a total derivative under a constant shift of B_{11}.

Since the fermionic isometry is assumed to be abelian (i.e. $Q^2 = 0$), one learns from the supersymmetry algebra

$$(\epsilon^\alpha Q_\alpha + \hat{\epsilon}^{\hat{\alpha}} Q_{\hat{\alpha}})^2 = (\epsilon \gamma^m \epsilon + \hat{\epsilon} \gamma^m \hat{\epsilon}) P_m \tag{2.30}$$

that

$$\epsilon \gamma^m \epsilon + \hat{\epsilon} \gamma^m \hat{\epsilon} = 0 \tag{2.31}$$

where $(Q_\alpha, Q_{\hat{\alpha}})$ are the supersymmetry generators and P_m is the translation generator. So (2.28) implies that $\partial_m C = 2i \epsilon \gamma_m \epsilon = -2i \hat{\epsilon} \gamma_m \hat{\epsilon}$. Note that if ϵ^α and $\hat{\epsilon}^{\hat{\alpha}}$ were Majorana spinors, (2.31) would imply that $\epsilon^\alpha = \hat{\epsilon}^{\hat{\alpha}} = 0$ since $(\gamma^0)_{\alpha\beta}$ is equal to the identity matrix in this basis. So the only non-trivial solutions to (2.31) involve complex Killing spinors ϵ^α and $\hat{\epsilon}^{\hat{\alpha}}$. In general, the T-duality transformation of (2.25) will therefore not map real background fields into real background fields.

2.5 Supersymmetry of T-dualized background

As was shown in the previous subsection, the fermionic T-duality transformation of (2.12) and (2.13) leaves invariant the component NS-NS fields $g_{mn}(x)$ and $b_{mn}(x)$, and transforms the Ramond-Ramond bispinor field-strength $F^{\alpha\hat{\beta}}(x)$ and dilaton $\phi(x)$ as

$$-\frac{i}{4} e^{\phi'} F'^{\alpha\hat{\beta}} = -\frac{i}{4} e^{\phi} F^{\alpha\hat{\beta}} - \epsilon^\alpha \hat{\epsilon}^{\hat{\beta}} C^{-1}, \quad \phi' = \phi + \frac{1}{2} \log C \tag{2.32}$$

where $C(x)$ is the $\theta = \hat{\theta} = 0$ component of B_{11} which satisfies

$$\partial_m C = 2i \epsilon \gamma_m \epsilon = -2i \hat{\epsilon} \gamma_m \hat{\epsilon}, \tag{2.33}$$

and $(\epsilon^\alpha(x), \hat{\epsilon}^{\hat{\alpha}}(x))$ are the Killing spinors associated to the fermionic shift isometry of (2.8). In other words, if one performs a local Type II supersymmetry transformation with Killing spinors $(\epsilon^\alpha(x), \hat{\epsilon}^{\hat{\alpha}}(x))$, the original background is assumed to be invariant.

A useful check of the transformations of (2.32) is that they should map a supersymmetric Type II background into a supersymmetric Type II background. If the original

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supersymmetry corresponding to a constant shift of θ^1 is described by Killing spinors $(\epsilon, \hat{\epsilon})$, the T-dualized supersymmetry corresponding to a constant shift of $\tilde{\theta}^1$ will be described by Killing spinors $\epsilon' = C^{-1}\epsilon$ and $\hat{\epsilon}' = C^{-1}\hat{\epsilon}$. One can also consider backgrounds with n abelian supersymmetries corresponding to constant shifts of θ^J for $J = 1$ to n. In this case, the n Killing spinors $(\epsilon^\alpha_J, \hat{\epsilon}^{\hat{\alpha}}_J)$ should satisfy the identities

$$\epsilon^\alpha_J \gamma^m_{\alpha\beta}\epsilon^\beta_K + \hat{\epsilon}^{\hat{\alpha}}_J \gamma^m_{\hat{\alpha}\hat{\beta}}\hat{\epsilon}^{\hat{\beta}}_K = \epsilon_J\gamma^m\epsilon_K + \hat{\epsilon}_J\gamma^m\hat{\epsilon}_K = 0 \tag{2.34}$$

for $J, K = 1$ to n so that the n supersymmetries anticommute with each other.

After performing T-duality with respect to θ^J for $J = 1$ to n, one finds that the Ramond-Ramond field-strength $F^{\alpha\hat{\beta}}(x)$ and dilaton $\phi(x)$ transform as

$$-\frac{i}{4}e^{\phi'}F'^{\alpha\hat{\beta}} = -\frac{i}{4}e^{\phi}F^{\alpha\hat{\beta}} - \epsilon^\alpha_J(C^{-1})_{JK}\hat{\epsilon}^{\hat{\beta}}_K, \quad \phi' = \phi + \frac{1}{2}\sum_{J=1}^{n}(\log C)_{JJ} \tag{2.35}$$

where $C_{JK}(x) = C_{KJ}(x)$ is the $\theta = \hat{\theta} = 0$ component of B_{JK} which satifies

$$\partial_m C_{JK} = 2i\epsilon_J\gamma_m\epsilon_K = -2i\hat{\epsilon}_J\gamma_m\hat{\epsilon}_K. \tag{2.36}$$

Furthermore, the new Killing spinors after performing T-duality are

$$\epsilon'^\alpha_J = (C^{-1})_{JK}\epsilon^\alpha_K, \quad \hat{\epsilon}'^{\hat{\alpha}}_J = (C^{-1})_{JK}\epsilon^{\hat{\alpha}}_K. \tag{2.37}$$

Under N=2 d=10 supersymmetry transformations parameterized by $(\rho_J\epsilon^\alpha_J, \rho_J\hat{\epsilon}^\alpha_J)$ where ρ_J are Grassmann constants, the dilatino λ_α and gravitino ψ^α_m transform in string frame as [27][3]

$$\begin{aligned}\delta_J\lambda_\alpha &= \partial_m\phi(\gamma^m\epsilon_J)_\alpha + 2i(\gamma_m P\gamma_m\hat{\epsilon}_J)_\alpha + \frac{1}{12}H_{mnp}(\gamma^{mnp}\epsilon_J)_\alpha, \\ \delta_J\psi^\alpha_m &= \nabla_m\epsilon^\alpha_J + 2i(P\gamma_m\hat{\epsilon}_J)^\alpha + \frac{1}{8}H_{mnp}(\gamma^{np}\epsilon_J)^\alpha,\end{aligned} \tag{2.38}$$

where $P^{\alpha\hat{\beta}} \equiv -\frac{i}{4}e^{\phi}F^{\alpha\hat{\beta}}$ and H_{mnp} is the Neveu-Schwarz three-form field-strength. After T-dualizing all fields and Killing spinors on the right-hand side of (2.38), one finds

$$\begin{aligned}\delta'_J\lambda'_\alpha &= \partial_m\phi'(\gamma^m\epsilon'_J)_\alpha + 2i(\gamma_m P'\gamma^m\hat{\epsilon}'_J)_\alpha + \frac{1}{8}H_{mnp}(\gamma^{mnp}\epsilon'_J)_\alpha \\ &= (C^{-1})_{JK}\delta_K\lambda_\alpha + \frac{1}{2}(C^{-1})_{KL}(\partial_m C)_{LK}(\gamma^m\epsilon_M)_\alpha(C^{-1})_{JM} \\ &\quad -2i(\gamma^m\epsilon_K)_\alpha(C^{-1})_{KL}(\hat{\epsilon}_L\gamma^m\hat{\epsilon}_M)(C^{-1})_{JM} \\ &= (C^{-1})_{JK}\delta_K\lambda_\alpha + i(C^{-1})_{KL}(\epsilon_L\gamma_m\epsilon_K)(\gamma^m\epsilon_M)_\alpha(C^{-1})_{JM} + \\ &\quad +2i(\gamma^m\epsilon_K)_\alpha(C^{-1})_{KL}(\epsilon_L\gamma^m\epsilon_M)(C^{-1})_{JM} = (C^{-1})_{JK}\delta_K\lambda_\alpha\end{aligned}$$

where we used the gamma-matrix identity

$$(\epsilon_L\gamma_m\epsilon_K)(\gamma^m\epsilon_J)_\alpha + (\epsilon_K\gamma_m\epsilon_J)(\gamma^m\epsilon_L)_\alpha + (\epsilon_J\gamma_m\epsilon_L)(\gamma^m\epsilon_K)_\alpha = 0 \tag{2.39}$$

[3] Our conventions differ from the ones in [27] by a factor of 4 for the RR fields. Namely, we have $P = -\frac{i}{4}e^{\phi}F^{\alpha\hat{\beta}}_{\text{ours}} = -\frac{i}{16}e^{\phi}F^{\alpha\hat{\beta}}_{[27]}$ where $F^{\alpha\hat{\beta}} = (\gamma^m)^{\alpha\hat{\beta}}F_m + \frac{1}{3!}(\gamma^{m_1m_2m_3})^{\alpha\hat{\beta}}F_{m_1m_2m_3} + \frac{1}{2}\frac{1}{5!}(\gamma^{m_1\cdots m_5})^{\alpha\hat{\beta}}F_{m_1\cdots m_5}$ in both cases.

We also have

$$
\begin{aligned}
\delta'_J \psi'^{\alpha}_m &= \nabla_m \epsilon'^{\alpha}_J + 2i(P'\gamma_m \hat{\epsilon}'_J)^{\alpha} + \frac{1}{12} H_{mnp}(\gamma^{np}\epsilon'_J)^{\alpha} \\
&= (C^{-1})_{JK}\delta_K \psi^{\alpha}_m - (C^{-1}(\partial_m C)C^{-1})_{JK}\epsilon^{\alpha}_K - 2i\epsilon^{\alpha}_K (C^{-1})_{KL}(\hat{\epsilon}_L \gamma_m \hat{\epsilon}_M)(C^{-1})_{JM} \\
&= (C^{-1})_{JK}\delta_K \psi^{\alpha}_m + 2i(C^{-1})_{JM}(\hat{\epsilon}_M \gamma_m \hat{\epsilon}_L)(C^{-1})_{LK}\epsilon^{\alpha}_K \\
&\qquad\qquad\qquad\qquad\quad -2i\epsilon^{\alpha}_K (C^{-1})_{KL}(\hat{\epsilon}_L \gamma_m \hat{\epsilon}_M)(C^{-1})_{JM} \\
&= (C^{-1})_{JK}\delta_K \psi^{\alpha}_m
\end{aligned} \tag{2.40}
$$

So if the background is supersymmetric before T-duality (i.e. if $\delta_J \lambda_\alpha = \delta_J \psi^{\alpha}_m = 0$), it is also supersymmetric after T-duality (i.e. $\delta'_J \lambda'_\alpha = \delta'_J \psi'^{\alpha}_m = 0$).

2.6 Null Ramond-Ramond field strength

The simplest example of fermionic T-duality is in a flat background where the supersymmetry parameters ϵ^α and $\hat{\epsilon}^{\hat{\beta}}$ are constants. One usually does not include the term $\int d^2 z B_{11} \partial\theta^1 \bar{\partial}\theta^1$ in the flat worldsheet action, but if B_{11} is constant, this term is a total derivative and can be included without affecting the equations of motion.

Since $\partial_m B_{11} = 0$, (2.33) implies that the supersymmetry parameters must be chosen to satisfy

$$
\epsilon\gamma^m \epsilon = \hat{\epsilon}\gamma^m \hat{\epsilon} = 0, \tag{2.41}
$$

i.e. ϵ^α and $\hat{\epsilon}^{\hat{\alpha}}$ are d=10 pure spinors. Since (2.41) has no Majorana-Weyl solutions in d=10 Minkowski space, one needs to consider complexified supersymmetry parameters.

After performing the T-duality transformations of (2.32), one finds that the dilaton shifts by a constant and the Ramond-Ramond field strength picks up the constant value

$$
e^{\phi'} F'^{\alpha\hat{\beta}} = 4i\epsilon^{\alpha}\hat{\epsilon}^{\hat{\beta}} C^{-1}. \tag{2.42}
$$

Since the stress tensor T^{mn} for a bispinor Ramond-Ramond field strength is proportional to $\gamma^m_{\alpha\beta}\gamma^n_{\hat{\gamma}\hat{\delta}} F^{\alpha\hat{\gamma}} F^{\beta\hat{\delta}}$ and since ϵ^α and $\hat{\epsilon}^{\hat{\beta}}$ are pure spinors satisfying (2.41), $F'^{\alpha\hat{\beta}}$ is a "null" bispinor which does not contribute to the stress-tensor and does not produce a back-reaction.

A closely related example which will be discussed in the following subsection arises as follows. One starts with a Calabi-Yau compactification to four dimensions which preserves N=2 d=4 supersymmetry, and one chooses ϵ^α and $\hat{\epsilon}^{\hat{\beta}}$ to be the chiral N=2 d=4 supersymmetry parameters. In this case, the resulting T-dualized background of (2.42) involves the self-dual graviphoton field-strength of [28 – 30] which leads to non-anti-commutative N=1 d=4 super-Yang-Mills on a D3 brane. As predicted by T-duality, the closed superstring spectrum in this self-dual graviphoton background is identical to the spectrum without the self-dual graviphoton field-strength. But this example shows clearly that fermionic T-duality changes the theory at higher loops since, unlike in a flat background, certain F terms in the effective action for a constant graviphoton background have been computed [31] and are non-zero in general.

2.7 Self-dual graviphoton background

To explicitly derive the T-duality transformations for the sigma model in a flat $d = 4$ background with Calabi-Yau compactification, it is convenient to use the $d = 4$ hybrid formalism for describing the worldsheet action. In a flat background, the worldsheet action is

$$S = \int d^2z[\partial x^{a\dot{a}}\bar{\partial} x_{a\dot{a}} + p_a\bar{\partial}\theta^a + \hat{p}_a\partial\hat{\theta}^a + \bar{p}_{\dot{a}}\bar{\partial}\bar{\theta}^{\dot{a}} + \hat{\bar{p}}_{\dot{a}}\partial\hat{\bar{\theta}}^{\dot{a}}] + S_C \tag{2.43}$$

where $a, \dot{a} = 1$ to 2 and S_C is the action for the compactified sector of the superstring. As discussed in [29], one can choose a chiral representation such that $q_a = \int dz p_a$ and $\hat{q}_a = \int d\bar{z}\hat{p}_a$ are the chiral spacetime supersymmetry generators. In this chiral representation, both the worldsheet action and the BRST operator are invariant under the shift isometries

$$\theta^a \to \theta^a + \rho^a, \quad \hat{\theta}^a \to \hat{\theta}^a + \hat{\rho}^a \tag{2.44}$$

where ρ^a and $\hat{\rho}^a$ are constants and all other worldsheet variables are unchanged.

After adding to (2.43) the surface term

$$\int d^2z C_{ab}[\partial\theta^a\bar{\partial}\hat{\theta}^b - \bar{\partial}\theta^a\partial\hat{\theta}^b] \tag{2.45}$$

where $C_{ab} = C_{ba}$ is a constant symmetric bispinor, one can T-dualize the shift isometries of (2.44) by introducing the fermionic gauge fields $(A^a, \bar{A}^a)$ and $(\hat{A}^a, \hat{\bar{A}}^a)$ to obtain the action

$$\begin{aligned} S = \int d^2z[&\partial x^{a\dot{a}}\bar{\partial} x_{a\dot{a}} + p_a\bar{A}^a + \hat{p}_a\hat{A}^a + C_{ab}(A^a\hat{\bar{A}}^b - \bar{A}^a\hat{A}^b) \\ &+\tilde{\theta}_a(\partial\bar{A}^a - \bar{\partial}A^a) + \hat{\tilde{\theta}}_a(\partial\hat{\bar{A}}^a - \bar{\partial}\hat{A}^a) + \bar{p}_{\dot{a}}\bar{\partial}\bar{\theta}^{\dot{a}} + \hat{\bar{p}}_{\dot{a}}\partial\hat{\bar{\theta}}^{\dot{a}}] + S_C. \end{aligned} \tag{2.46}$$

Integrating out the worldsheet gauge fields produces a constant shift of the dilaton and the worldsheet action becomes

$$\begin{aligned} S = \int d^2z[&\partial x^{a\dot{a}}\bar{\partial} x_{a\dot{a}} + (C^{-1})^{ab}(p_a\bar{\partial}\hat{\tilde{\theta}}_b + \hat{p}_a\partial\tilde{\theta}_b + p_a\hat{p}_b - \partial\tilde{\theta}_a\bar{\partial}\hat{\tilde{\theta}}_b + \bar{\partial}\tilde{\theta}_a\partial\hat{\tilde{\theta}}_b) \\ &+\bar{p}_{\dot{a}}\bar{\partial}\bar{\theta}^{\dot{a}} + \hat{\bar{p}}_{\dot{a}}\partial\hat{\bar{\theta}}^{\dot{a}}] + S_C. \end{aligned} \tag{2.47}$$

After dropping the surface term $\int d^2z(C^{-1})^{ab}(\partial\tilde{\theta}_a\bar{\partial}\hat{\tilde{\theta}}_b + \bar{\partial}\tilde{\theta}_a\partial\hat{\tilde{\theta}}_b)$ and defining $\phi^a = (C^{-1})^{ab}\hat{\tilde{\theta}}_b$ and $\hat{\phi}^a = (C^{-1})^{ab}\tilde{\theta}_b$, one obtains the action

$$S = \int d^2z[\partial x^{a\dot{a}}\bar{\partial} x_{a\dot{a}} + p_a\bar{\partial}\phi^a + \hat{p}_a\partial\hat{\phi}^a + (C^{-1})^{ab}p_a\hat{p}_b + \bar{p}_{\dot{a}}\bar{\partial}\bar{\theta}^{\dot{a}} + \hat{\bar{p}}_{\dot{a}}\partial\hat{\bar{\theta}}^{\dot{a}}] + S_C, \tag{2.48}$$

which is the worldsheet action of [28–30] in a background with constant self-dual field-strength F^{ab} proportional to $e^{-\phi}(C^{-1})^{ab}$.

The difference between loop amplitudes in the constant self-dual graviphoton background and loop amplitudes in a flat background comes from the presence of the term $(C^{-1})^{ab}\int d^2z p_a\hat{p}_b$ in the self-dual graviphoton worldsheet action. Since p_a is holomorphic and $\hat{p}_a$ is antiholomorphic, this term can be written as $(C^{-1})^{ab}(\int dz p_a)(\int d\bar{z}\hat{p}_b)$ where the contours of $\int dz$ and $\int d\bar{z}$ go around the non-trivial cycles of the genus g surface. So this term can absorb the higher-genus zero modes associated with the fermionic one-forms p_a and $\hat{p}_a$. Again, higher genus amplitudes are sensitive to the presence of the constant graviphoton field strength [31] . This shows that fermionic T-duality is not a full symmetry of the theory at higher genus.

3. Exact T-duality of the $AdS_5 \times S^5$ background

In this section, we show that after performing bosonic T-duality with respect to the $d = 4$ coordinates (x^0, x^1, x^2, x^3) and performing fermionic T-duality with respect to 8 of the 32 fermionic coordinates $\theta^{\alpha j}$, the original $AdS_5 \times S^5$ background is mapped to another $AdS_5 \times S^5$ background with constant dilaton. The transformation is an exact change of variables in the path integral, with a unit jacobian. Thus this is an exact symmetry to all orders in the α' expansion and it is also expected to be an exact symmetry non-perturbatively in α'. We will first show this by analyzing the transformations of the $AdS_5 \times S^5$ background fields, and we will then show it again by explicitly T-dualizing the Green-Schwarz and pure spinor versions of the $AdS_5 \times S^5$ sigma model.

3.1 T-duality transformations of the $AdS_5 \times S^5$ background fields

A non-trivial example of fermionic T-duality arises in the $AdS_5 \times S^5$ background which has 32 fermionic isometries. These isometries can be identified with the N=4 d=4 supersymmetry transformations $(q^{aj}, \bar{q}^{\dot{a}}_j)$ and the N=4 d=4 superconformal transformations $(s^a_j, \bar{s}^{\dot{a}j})$, and one can choose 8 of the 32 fermionic symmetries to anticommute with each other and to also commute with the four $d = 4$ translations. A convenient choice for the abelian subset are the 8 chiral supersymmetry generators q^{aj} which will be associated with the Killing spinors $(\epsilon^\alpha_{aj}, \hat{\epsilon}^{\hat{\alpha}}_{aj})$. After T-dualizing with respect to these 8 abelian fermionic isometries, (2.35) implies that

$$-\frac{i}{4} e^{\phi'} F'^{\alpha\hat{\beta}} = -\frac{i}{4} e^{\phi} F^{\alpha\hat{\beta}} - \epsilon^\alpha_{aj} \hat{\epsilon}^{\hat{\beta}}_{bk} (C^{-1})^{aj\ bk}, \quad \phi' = \phi + \frac{1}{2} \mathrm{Tr} \log C \tag{3.1}$$

We can determine C in two ways. We could use the explicit form of the Killing spinors and use (2.36), or we could view $C_{aj\ bk}$ as the $\theta = \hat{\theta} = 0$ component of $B_{aj\ bk}$. We will follow this second route.

In an $AdS_5 \times S^5$ background, one can choose a gauge where the only nonzero components of $B_{AB} \equiv E^M_A E^N_B B_{MN}$ are the components

$$B_{\alpha\hat{\beta}} = B_{\hat{\beta}\alpha} = -i(\gamma^{01234})_{\alpha\hat{\beta}}. \tag{3.2}$$

This gauge choice is not possible in a flat background, and it simplifies the Green-Schwarz Wess-Zumino term in an $AdS_5 \times S^5$ background [32]. In this gauge, $C_{aj\ bk}$ is the $\theta = \hat{\theta} = 0$ component of $\epsilon^\alpha_{aj} \hat{\epsilon}^{\hat{\beta}}_{bk} B_{\alpha\hat{\beta}} + \epsilon^{\hat{\alpha}}_{aj} \hat{\epsilon}^{\beta}_{bk} B_{\hat{\alpha}\beta}$, so one finds that

$$C_{aj\ bk} = -2i \epsilon^\alpha_{aj} (\gamma^{01234})_{\alpha\hat{\beta}} \hat{\epsilon}^{\hat{\beta}}_{bk}. \tag{3.3}$$

It is convenient to write the $AdS_5 \times S^5$ metric as

$$ds^2 = |y|^{-2}(dx^m dx_m + dy_r dy_r) \tag{3.4}$$

where $\frac{y_r}{|y|}$ for $r = 1$ to 6 are the variables on S^5 and $|y|$ is the fifth variable on AdS_5. It is also convenient to decompose the local spinor indices α, $\hat{\alpha}$ into $\mathrm{SO}(3,1) \times \mathrm{SO}(5)$ as $\alpha = (a'j', \dot{a}'j')$. Note that $j' = 1, \cdots, 4$ is an SO(5) spinor index that can be raised and lowered using $(\sigma^6)_{j'k'}$ where $(\sigma^r)_{j'k'}$ are the SO(6) Pauli matrices. In terms of this decomposition we have that $(\gamma^{01234})^{a'j'\,b'k'} = i\epsilon^{a'b'}(\sigma^6)^{j'k'}$. In order to write the form of the Killing spinors we introduce the rotation matrix $M_j^{k'}(y)$ which is the $\frac{\mathrm{SU}(4)}{\mathrm{SO}(5)}$ matrix which rotates the point $(0,0,0,0,0,1)$ on S^5 to the point $|y|^{-1}(y_1, y_2, y_3, y_4, y_5, y_6)$. The Killing spinors $\epsilon_{aj}{}^{\alpha}$ and $\hat{\epsilon}_{aj}{}^{\hat{\alpha}}$ can be written as

$$\epsilon_{aj}{}^{b'k'} = |y|^{\frac{1}{2}}\delta_a^{b'} M_j^{k'}(y), \quad \epsilon_{aj}{}^{\dot{b}k'} = 0, \quad \hat{\epsilon}_{aj}{}^{b'k'} = i|y|^{\frac{1}{2}}\delta_a^{b'} M_j^{k'}(y), \quad \hat{\epsilon}_{aj}^{\dot{b}'k'} = 0. \tag{3.5}$$

Using (3.3) and the identity

$$M_l^{j'}(\gamma^{01234})_{a'j'\ b'k'} M_m^{k'} = i\epsilon_{a'b'}(\sigma^r)_{lm}|y|^{-1}y_r, \tag{3.6}$$

one finds that $C_{aj\ bk} = 2i\epsilon_{ab}\sigma^r_{jk}y_r$ and $(C^{-1})^{aj\ bk} = -\frac{i}{2}\epsilon^{ab}(\sigma^r)^{jk}\frac{y_r}{|y|^2}$. This formula for C obeys equation (2.36) . In fact, we could have simply derived the expression for C by solving (2.36) . To determine the transformation of $F^{\alpha\hat{\beta}}$ in (3.1), note that

$$\epsilon_{aj}^{a'j'}(C^{-1})^{aj\ bk}\hat{\epsilon}_{bk}^{b'k'} = \frac{1}{2}\epsilon^{a'b'}(\sigma^6)^{j'k'} = -\frac{i}{2}(\gamma_{01234})^{a'j'\ b'k'}. \tag{3.7}$$

Note that we get the projection of the matrix γ^{01234} to the part with definite four dimensional chirality. Thus, we can write it in terms of a projection operator $\frac{1}{2}[(\gamma_{0123} - i)\gamma_4]^{\alpha\hat{\beta}}$. This only has nonzero components when $\alpha = a'j'$ and $\hat{\beta} = b'k'$ so that $\frac{1}{2}[(\gamma_{0123} - i)\gamma_4]^{a'j'\ b'k'} = (\gamma_{01234})^{a'j'\ b'k'}$, and one finds that

$$\begin{aligned} e^{\phi'}F'^{\alpha\hat{\beta}} &= e^{\phi}F^{\alpha\hat{\beta}} - 4i\epsilon^{\alpha}_{aj}\hat{\epsilon}^{\hat{\beta}}_{bk}(C^{-1})^{aj\ bk} \\ &= (\gamma_{01234})^{\alpha\hat{\beta}} - (\gamma_{01234} - i\gamma_4)^{\alpha\hat{\beta}} = (i\gamma_4)^{\alpha\hat{\beta}} \end{aligned} \tag{3.8}$$

where the γ-matrices appearing in (3.8) have tangent-space vector indices. The dual background therefore has an imaginary RR scalar field which varies only along the radial AdS direction. Also, $Tr(\log C) = 8\log|y|$ implies that

$$\phi' = \phi + 4\log|y|. \tag{3.9}$$

If one now T-dualizes with respect to the four translation symmetries of x^m, it is easy to verify that the one-form Ramond-Ramond field strength proportional to $(i\gamma_4)^{\alpha\hat{\beta}}$ transforms back into the five-form Ramond-Ramond field strength proportional to $(\gamma_{01234})^{\alpha\hat{\beta}}$, and that the dilaton shifts back to $\phi = \phi' - 4\log|y|$. Note that the factor of i in front of γ^4 disappears again in Minkowski space when we T-dualize along the time direction x^0 [33] . So the $AdS_5 \times S^5$ background fields are invariant under the combined bosonic and fermionic T-duality transformations.

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It is interesting to note that there is another combination of bosonic and fermionic T-duality transformations which also leaves the $AdS_5 \times S^5$ background invariant. If one breaks SU(4) R-symmetry to $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ by choosing a U(1) direction in the SU(4), the SU(4) index $j = 1$ to 4 splits into an index $r = 1$ to 2 which carries $+1$ charge with respect to the chosen U(1) direction, and an index $r' = 3$ to 4 which carries -1 U(1) charge. Under this subgroup of SU(4), the 32 supersymmetries split into $(q^a_r, q^a_{r'}, \bar{q}^r_{\dot{a}}, \bar{q}^{r'}_{\dot{a}})$ and $(s^r_a, s^{r'}_a, \bar{s}^{\dot{a}}_r, \bar{s}^{\dot{a}}_{r'})$, and one can choose the 8 abelian supersymmetries to be $q^a_{r'}$ and $\bar{q}^r_{\dot{a}}$ which all carry $+1$ U(1) charge. Furthermore, under the breakup of SU(4) into $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$, the SU(4) generators R^k_j split into $(R^s_r, R^{s'}_{r'}, R^{s'}_r, R^s_{r'})$ where the four generators $R^s_{r'}$ all carry $+2$ U(1) charge. Together with the four translations of (x^0, x^1, x^2, x^3), the 8 supersymmetries $(q^a_{r'}, \bar{q}^r_{\dot{a}})$ and 4 SU(4) transformations $R^s_{r'}$ form an abelian subgroup of $PSU(2,2|4)$ isometries with 8 bosonic and 8 fermionic generators. After performing T-duality with respect to these 8 bosonic and 8 fermionic isometries, one finds using a similar analysis as above that the $AdS_5 \times S^5$ background is invariant.

Note that the translation generators $R^s_{r'}$ that we chose in the five-sphere are not hermitian, so this choice will involve a complexification of the coordinates. An alternative way to see this is to do an analytic continuation of the S^5 coordinates into dS^5 (five dimensional de-Sitter space) and then write the metric of dS^5 as $ds^2 = \frac{-dw^2 + du_i du_i}{w^2}$. With this choice, the four translation symmetries shift the four u coordinates.

This alternative choice of abelian isometries is related to harmonic $\mathcal{N} = 4$ $d = 4$ superfields in the same way that the previous choice of abelian isometries using q_{aj} is related to chiral $\mathcal{N} = 4$ $d = 4$ superfields. As discussed in [34] and [35], harmonic $\mathcal{N} = 4$ $d = 4$ superfields are naturally constructed using the supercoset $\frac{PSU(2,2|4)}{PS(\mathrm{U}(2|2)\times\mathrm{U}(2|2))}$ where the denominator $PS(\mathrm{U}(2|2) \times \mathrm{U}(2|2))$ consists of the generators

$$[M^b_a, M^{\dot{b}}_{\dot{a}}, D, R^s_r, R^{s'}_{r'}, q^a_r, \bar{q}^{r'}_{\dot{a}}, s^r_a, \bar{s}^{\dot{a}}_{r'}]. \tag{3.10}$$

The 16 bosonic and 16 fermionic generators in the supercoset $\frac{PSU(2,2|4)}{PS(\mathrm{U}(2|2)\times\mathrm{U}(2|2))}$ split into "upper-triangular" generators $[P^a_{\dot{a}}, R^s_{r'}, q^a_{r'}, \bar{q}^r_{\dot{a}}]$ and "lower-triangular" generators $[K^{\dot{a}}_a, R^{r'}_s, s^{r'}_a, \bar{s}^{\dot{a}}_r]$, and the "upper-triangular" generators are precisely the 8+8 abelian isometries which are T-dualized in this approach. This is closely related to the decomposition of $PSU(2,2|4)$ that one performs when we consider a 1/2 BPS string state with large charge (corresponding to an operator $Tr[Z^J]$). The upper vs. lower triangular generators act as creation vs annihilation operators for impurities along the string.

3.2 Invariance of $AdS_5 \times S^5$ Green-Schwarz sigma model

This invariance under the combined fermionic and bosonic T-dualities can also be verified by explicitly performing the T-duality transformations on the $AdS_5 \times S^5$ sigma model. To show this invariance, we will first consider the Green-Schwarz version of the sigma model and will then consider the pure spinor version.

In an $AdS_5 \times S^5$ background, the Green-Schwarz sigma model $S = \int d^2z[(G_{MN}(Z) + B_{MN}(Z))\partial Z^M \bar{\partial} Z^N]$ takes the form

$$S = \frac{R^2}{4\pi\alpha'} \int d^2z[\eta_{cd} J^c \bar{J}^d - i(\gamma^{01234})_{\alpha\hat{\beta}}(J^\alpha \bar{J}^{\hat{\beta}} - \bar{J}^\alpha J^{\hat{\beta}})] \tag{3.11}$$

where R is the AdS radius, $J^C = (g^{-1}\partial g)^C$ and $\bar{J}^C = (g^{-1}\bar{\partial} g)^C$ are left-invariant Metsaev-Tseytlin [36] currents constructed from the supercoset $g(Z) \in \frac{PSU(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$, and $C = (c, \alpha, \hat{\alpha})$ labels the $PSU(2,2|4)$ Lie-algebra generators which are not in $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$. More precisely, $c = 0$ to 4 labels the five AdS_5 generators of $\frac{\mathrm{SO}(4,2)}{\mathrm{SO}(4,1)}$, $c = 5$ to 9 labels the five S^5 generators of $\frac{\mathrm{SO}(6)}{\mathrm{SO}(5)}$, $\alpha = 1$ to 16 labels the supersymmetries originating from the "left-moving" half of the $\mathcal{N} = 2$ d=10 supersymmetry, and $\hat{\alpha} = 1$ to 16 labels the supersymmetries originating from the "right-moving" half.

Splitting the $\mathrm{SO}(9,1)$ indices into $\mathrm{SO}(3,1)\times\mathrm{SO}(5)$ indices, this action can be expressed as

$$S = \frac{R^2}{4\pi\alpha'}\int d^2z[(J_{P_m} + J_{K_m})(\bar{J}_{P_m} + \bar{J}_{K_m}) + J_D\bar{J}_D + \\ + J_{R_t}\bar{J}_{R_t} + J_{q^a_j}\bar{J}_{q^a_j} + J_{\bar{q}^j_{\dot{a}}}\bar{J}_{\bar{q}^j_{\dot{a}}} + J_{s^j_a}\bar{J}_{s^j_a} + J_{\bar{s}^{\dot{a}}_j}\bar{J}_{\bar{s}^{\dot{a}}_j}] \tag{3.12}$$

where P_m and K_m for $m = 0$ to 3 label the translations and conformal boosts, D labels the dilatations, R_t for $t = 1$ to 5 label the $\mathrm{SO}(6)/\mathrm{SO}(5)$ generators, and $(q^a_j, \bar{q}^j_{\dot{a}}, s^j_a, \bar{s}^a_j)$ label the fermionic supersymmetry and superconformal generators. Note that when written in terms of $\mathrm{SO}(3,1)\times\mathrm{SO}(5)$ spinor indices, the $(\gamma^{01234})_{\alpha\hat{\beta}}$ matrix in (3.11) decomposes as $(\gamma^{01234})_{aj\ bk} = i\epsilon_{ab}(\sigma^6)_{jk}$ and $(\gamma^{01234})_{\dot{a}j\ \dot{b}k} = i\epsilon_{\dot{a}\dot{b}}(\sigma^6)_{jk}$. So the a and $\dot{a}$ indices in (3.12) are contracted with ϵ_{ab} and $\epsilon_{\dot{a}\dot{b}}$, while the j indices are contracted with $(\sigma^6)_{jk}$.

To compute the transformation of (3.12) under T-duality, it is convenient to use the parameterization of the supercoset $g(Z)$ in which

$$g(x^m, y^t, \theta^{aj}, \bar{\theta}^{\dot{a}}_j, \bar{\xi}^j_{\dot{a}}) = \exp(x^m P_m + \theta^{aj}q_{aj})\exp(\bar{\theta}^{\dot{a}}_j\bar{q}^j_{\dot{a}} + \bar{\xi}^j_{\dot{a}}\bar{s}^{\dot{a}}_j)\ |y|^D\ \exp\left(\sum_{t=1}^{5}\frac{y^t}{|y|}R_t\right) \tag{3.13}$$

where $|y| = \sqrt{\sum_{t=1}^{6} y_t y_t}$ and $\frac{y_t}{|y|}$ for $t = 1$ to 6 are the S^5 coordinates. In this parameterization of g, κ-symmetry has been used to gauge-fix to zero the eight fermionic parameters associated with the $\mathcal{N} = 4$ d=4 chiral superconformal generators s^{aj}. But there are still eight remaining κ-symmetries which have not been gauge-fixed.

If one writes $g = \exp(x^m P_m + \theta^{aj}q_{aj})e^B$ where

$$e^B = \exp(\bar{\theta}^{\dot{a}}_j\bar{q}^j_{\dot{a}} + \bar{\xi}^j_{\dot{a}}\bar{s}^{\dot{a}}_j)\ |y|^D\ \exp\left(\sum_{t=1}^{5}\frac{y^t}{|y|}R_t\right), \tag{3.14}$$

then the left-invariant currents $g^{-1}\partial g$ take the form

$$\begin{aligned} J_{P_m} &= [e^{-B}(\partial x^n P_n + \partial\theta^{aj}q_{aj})e^B]_{P_m}, & J_{q^a_j} &= [e^{-B}(\partial x^m P_m + \partial\theta^{bk}q_{bk})e^B]_{q^a_j} \\ J_D &= [e^{-B}\partial e^B]_D, \quad J_{R_t} = [e^{-B}\partial e^B]_{R_t}, & J_{\bar{q}^j_{\dot{a}}} &= [e^{-B}\partial e^B]_{\bar{q}^j_{\dot{a}}}, \quad J_{\bar{s}^{\dot{a}}_j} = [e^{-B}\partial e^B]_{\bar{s}^{\dot{a}}_j}, \\ J_{K_m} &= 0, & J_{s^j_a} &= 0, \end{aligned} \tag{3.15}$$

where $[\ \ ]_I$ denotes the component of $[\ \ ]$ which is proportional to the Lie-algebra generator I. To understand the structure of (3.15), it is useful to note that the generators $(\bar{q}^j_{\dot{a}}, \bar{s}^{\dot{a}}_j, D, R^k_j, M_{\dot{a}\dot{b}})$ form an $\mathrm{SU}(2|4)$ supergroup where $M_{\dot{a}\dot{b}}$ are the anti-self-dual Lorentz generators. Under this SU(2|4) supergroup, the generators $(P_{a\dot{a}}, q_{aj})$ transform as a fundamental representation and the generators $(K^{a\dot{a}}, s^{aj})$ transform as an anti-fundamental representation.

One can now T-dualize with respect to x^m and θ^{aj} by introducing the bosonic gauge fields $(A^m, \bar{A}^m)$ and the fermionic gauge fields $(A^{aj}, \bar{A}^{aj})$, and adding the Lagrange multiplier term

$$\frac{R^2}{4\pi\alpha'}\int d^2z[\tilde{x}_m(\bar{\partial}A^m - \partial\bar{A}^m) + \tilde{\theta}_{aj}(\bar{\partial}A^{aj} - \partial\bar{A}^{aj})] \tag{3.16}$$

to the action of (3.12). The action then takes the form

$$S = \frac{R^2}{4\pi\alpha'}\int d^2z[A'^m\bar{A}'^m + A'^{aj}A'^{aj} + \tilde{x}_m(\bar{\partial}A^m - \partial\bar{A}^m) + \tilde{\theta}_{aj}(\bar{\partial}A^{aj} - \partial\bar{A}^{aj}) + \\ + J_D\bar{J}_D + J_{R_t}\bar{J}_{R_t} + J_{\bar{q}^j_{\dot{a}}}\bar{J}_{\bar{q}^j_{\dot{a}}} + J_{\bar{s}^{\dot{a}}_j}\bar{J}_{\bar{s}^{\dot{a}}_j}] \tag{3.17}$$

where

$$A'^m = [e^{-B}(A^nP_n + A^{aj}q_{aj})e^B]_{P_m}, \quad A'^{aj} = [e^{-B}(A^mP_m + A^{bk}q_{bk})e^B]_{q_{aj}}. \tag{3.18}$$

Writing $A^m = [e^B(A'^nP_n + A'^{aj}q_{aj})e^{-B}]_{P_m}$ and $A^{aj} = [e^B(A'^nP_n + A'^{bk}q_{bk})e^{-B}]_{q_{aj}}$ and integrating out A'^m and A'^{aj}, one finds that the T-dualized action is

$$S = \frac{R^2}{4\pi\alpha'}\int d^2z[J'_{P_m}\bar{J}'_{P_m} + J'_{q_{aj}}\bar{J}'_{q_{aj}} + J_D\bar{J}_D + J_{R_t}\bar{J}_{R_t} + J_{\bar{q}^j_{\dot{a}}}\bar{J}_{\bar{q}^j_{\dot{a}}} + J_{\bar{s}^{\dot{a}}_j}\bar{J}_{\bar{s}^{\dot{a}}_j}] \tag{3.19}$$

where $J'_{P_m} = [e^B(\partial\tilde{x}_nP_m)e^{-B}]_{P_n} + [e^B(\partial\tilde{\theta}_{aj}P_m)e^{-B}]_{q_{aj}}$ and $J'_{q_{aj}} = [e^B(\partial\tilde{x}_nq_{aj})e^{-B}]_{P_n} + [e^B(\partial\tilde{\theta}_{bk}q_{aj})e^{-B}]_{q_{bk}}$.

The integration over A' and $\bar{A}'$ gives a measure factor proportional to the superdeterminant of $|\frac{\partial A'}{\partial A}|$. Since B is an element of SU(2|4), the super-Jacobian in the transformation of (3.18) is equal to one. For example, if one restricts to the dilatation transformation parameterized by $|y|$, $A'^m = |y|A^m$ and $A'^{aj} = |y|^{\frac{1}{2}}A^{aj}$. Since there are four A^m's and eight A^{aj}'s, the super-Jacobian cancels. So the measure factor is equal to one which implies that the dilaton does not transform under the combined bosonic and fermionic T-duality.

To relate (3.19) to the original action of (3.12), note that

$$J'_{P_m} = Tr[e^B(\partial\tilde{x}_nP_m)e^{-B}K^n + e^B(\partial\tilde{\theta}_{aj}P_m)e^{-B}s^{aj}] = [e^{-B}(\partial\tilde{x}_nK^n + \partial\tilde{\theta}_{aj}s^{aj})e^B]_{K^m} \tag{3.20}$$

where Tr denotes the trace over $PSU(2,2|4)$ indices defined such that $Tr(P_mK^n) = \delta^n_m$ and $Tr(q_{aj}s^{bk}) = \delta^a_b\delta^j_k$. Similarly,

$$J'_{q_{aj}} = Tr[e^B(\partial\tilde{x}_nq_{aj})e^{-B}K^n + e^B(\partial\tilde{\theta}_{bk}q_{aj})e^{-B}s^{bk}] = [e^{-B}(\partial\tilde{x}_nK^n + \partial\tilde{\theta}_{bk}s^{bk})e^B]_{s^{aj}}. \tag{3.21}$$

Suppose one parameterizes

$$g(\tilde{x}, \tilde{\theta}, y, \bar{\theta}, \bar{\xi}) = \exp(\tilde{x}_mK^m + \tilde{\theta}_{aj}s^{aj})e^B \tag{3.22}$$

where e^B is defined as in (3.14) and κ-symmetry has been used to gauge-fix to zero the eight fermionic parameters associated with q_{aj}. Then the left-invariant currents $g^{-1}\partial g$ now take the form

$$\begin{aligned}
J_{K^m} &= [e^{-B}(\partial\tilde{x}_n K^n + \partial\tilde{\theta}_{aj}s^{aj})e^B]_{K^m}, & J_{s^{aj}} &= [e^{-B}(\partial\tilde{x}_m K^m + \partial\tilde{\theta}_{bk}s^{bk})e^B]_{s^{aj}},\\
J_D = [e^{-B}\partial e^B]_D, &\quad J_{R_t} = [e^{-B}\partial e^B]_{R_t}, & J_{\bar{q}^j_{\dot{a}}} = [e^{-B}\partial e^B]_{\bar{q}^j_{\dot{a}}}, &\quad J_{\bar{s}^{\dot{a}}_j} = [e^{-B}\partial e^B]_{\bar{s}^{\dot{a}}_j},\\
J_{P_m} &= 0, & J_{q_{aj}} &= 0.
\end{aligned} \tag{3.23}$$

So the T-dualized action of (3.19) reproduces the action of (3.12) if one uses the parameterization of (3.22).

Finally, one can relate the parameterization of (3.22) with the original parameterization of (3.13) by using the isomorphism of $PSU(2,2|4)$ which switches

$$P_m \to K^m, \quad q_{aj} \to s^{aj}, \quad \bar{q}^j_{\dot{a}} \to \bar{s}^{\dot{a}}_j, \quad D \to -D. \tag{3.24}$$

If one simultaneously switches the variables

$$x^m \to \tilde{x}_m, \quad \theta^{aj} \to \tilde{\theta}_{aj}, \quad \bar{\theta}^{\dot{a}}_j \to \bar{\xi}^j_{\dot{a}}, \quad y_t \to \frac{y_t}{|y|^2}, \tag{3.25}$$

the parameterization of (3.22) is mapped to the parameterization of (3.13). So it has been verified that after partially gauge-fixing the κ-symmetry, the Green-Schwarz version of the $AdS_5 \times S^5$ sigma model is mapped to itself under the combined T-duality with respect to x^m and θ^{aj}.

Since the argument above might have been too detailed, let us repeat the gist of the argument using SU(2|4) invariant notation. We group the coordinates as $Z^{aI} = (x^{a\dot{a}}, \theta^{aj})$ where I is an SU(2|4) index. We also have the corresponding generators $G_{aI} = (P_{a\dot{a}}, q_{aj})$ and their dual generators $G^{aI} = (K^{a\dot{a}}, s^{aj})$. We can then write the part of the action depending on Z^{aj} as

$$S \sim \int \epsilon_{ab}\eta_{IJ} M^I{}_L M^J{}_K \partial Z^{aI} \bar{\partial} Z^{bK} \tag{3.26}$$

where η_{IJ} is the supergroup invariant metric and $M^I{}_L$ is given by

$$Tr[G^{aI}e^{-B}G_{bL}e^B] = \delta^a_b M^I{}_L\,, \qquad Tr[G^{aI}G_{bL}] = \delta^a_b\delta^I_L \tag{3.27}$$

After the T-duality we end up with dual variables $\widetilde{Z}_{aj}$ and the action will be of a similar form but it will involve the inverse of this matrix. This inverse can be written by an expression similar to (3.27) but involving the inverse transformation

$$\delta^a_b (M^{-1})^I{}_L = Tr[G^{aI}e^B G_{bL}e^{-B}] = Tr[e^{-B}G^{aI}e^B G_{bL}] \tag{3.28}$$

where in the last expression we noticed that the inverse matrix can be viewed as the same transformation e^B as in (3.27) but acting on the dual generators G^{aJ}. Thus, after performing the transformation that exchanges the dual generators with the original ones we end up with the same form of the action.

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At the end of the previous subsection, we discussed an alternative choice of T-dualization which also leaves the $AdS_5 \times S^5$ background invariant. One can show invariance of the Green-Schwarz sigma model using this alternative T-dualization by replacing the above SU(2|4) subgroup of $PSU(2,2|4)$ with the $PS(\mathrm{U}(2|2) \times \mathrm{U}(2|2))$ subgroup of (3.10). After gauge-fixing to zero the 8 fermionic parameters associated with $s_a^{r'}$ and $\bar{s}_r^{\dot{a}}$, one can follow the same steps as above. One first groups the coordinates as $Z_I^{J'} = (x_a^{\dot{a}}, u_r^{r'}, \theta_a^{r'}, \bar{\theta}_r^{\dot{a}})$ where $I = (a, r)$ and $J' = (\dot{a}, r')$ are $\mathrm{U}(2|2) \times \mathrm{U}(2|2)$ indices, and $u_r^{r'}$ are four coordinates on the (analytic continuation of) S^5. The corresponding generators are $G_{J'}^I = (P_{\dot{a}}^a, R_{r'}^r, q_{r'}^a, \bar{q}_{\dot{a}}^r)$ and their dual generators are $G_I^{J'} = (K_a^{\dot{a}}, R_r^{r'}, s_a^{r'}, \bar{s}_r^{\dot{a}})$. One can now repeat the procedures of (3.26) - (3.28) to show that the action is mapped to itself under this T-duality.

3.3 Invariance of the $AdS_5 \times S^5$ pure spinor sigma model

In the previous subsection, it was shown that in the gauge $\xi_{aj} = 0$, the Green-Schwarz version of the $AdS_5 \times S^5$ action is invariant under T-duality where ξ_{aj} correspond to the 8 fermionic parameters asssociated with the chiral superconformal generators s^{aj}. In other words, the general element of the $\frac{PSU(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$ coset is

$$\tilde{g} = g(x, y, \theta, \bar{\theta}, \bar{\xi}) \exp(\xi_{aj} s^{aj}) \tag{3.29}$$

where $g(x, y, \theta, \bar{\theta}, \bar{\xi})$ is the gauge-fixed supercoset used in (3.13). It will now be shown that the pure spinor version of the $AdS_5 \times S^5$ action is also invariant under T-duality. Since the pure spinor version of the action is quantizable, this proves that the sigma model action in an $AdS_5 \times S^5$ background is invariant under T-duality to all orders in α'.

The first step is to use the fact that there is a unique prescription for constructing the pure spinor action from any κ-invariant Green-Schwarz action. This prescription was first described by Oda and Tonin [37] and involves relating the Green-Schwarz κ-transformations with the pure spinor BRST transformations. So if the T-dualized Green-Schwarz action could be written in a κ-invariant form, one could use this prescription to prove that T-dualization does not change the pure spinor action.

However, the T-dualized Green-Schwarz action was only shown to be invariant in the gauge $\xi_{aj} = 0$. This means that the original and T-dualized pure spinor actions may differ by terms which vanish when $\xi_{aj} = 0$. It will now be argued using BRST invariance that such terms cannot be present. Note that invariance under T-duality of the BRST operators $Q = \int dz \lambda^\alpha d_\alpha$ and $\hat{Q} = \int d\bar{z} \hat{\lambda}^{\hat{\alpha}} \hat{d}_{\hat{\alpha}}$ is manifest since the worldsheet variables $(\lambda^\alpha, \hat{\lambda}^{\hat{\alpha}})$ and $(d_\alpha, \hat{d}_{\hat{\alpha}})$ transform by local $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ Lorentz rotations under T-duality.

Suppose that the original pure spinor action is S_0 and the T-dualized pure spinor action is S_1 where

$$S_1 = S_0 + \int d^2z \; \xi_{aj} V^{aj} \tag{3.30}$$

for some V^{aj}. Then BRST invariance of S_0 and S_1 implies that

$$\int d^2z \; (Q + \hat{Q})(\xi_{aj} V^{aj}) = 0. \tag{3.31}$$

Furthermore, as explained in [38], Q and $\hat{Q}$ act on the supercoset element $\tilde{g}$ of (3.29) by right multiplication as

$$(Q+\hat{Q})\tilde{g} = \tilde{g}[(\lambda^{aj}+\hat{\lambda}^{aj})q_{aj} + (\lambda_{aj}-\hat{\lambda}_{aj})s^{aj} + (\lambda^{\dot{a}}_{j}-\hat{\lambda}^{\dot{a}}_{j})q^{j}_{\dot{a}} + (\lambda^{j}_{\dot{a}}+\hat{\lambda}^{j}_{\dot{a}})s^{\dot{a}}_{j})] \tag{3.32}$$

where the j indices on λ^{aj} and $\hat{\lambda}^{aj}$ are SO(5) spinor indices which can be raised and lowered using $(\sigma^6)_{jk}$ and $(\sigma^6)^{jk}$. Using (3.29) and (3.32), one learns that the only worldsheet field which transforms into $(\lambda_{aj}-\hat{\lambda}_{aj})$ is ξ_{aj} which has the BRST transformation $(Q+\hat{Q})\xi_{aj} = (\lambda_{aj}-\hat{\lambda}_{aj})+\ldots$ where the terms in $\ldots$ will not concern us.

Suppose one expands

$$V^{a_1j_1} = V^{a_1j_1}_{(1)} + \xi_{a_2j_2}V^{a_1j_1\ a_2j_2}_{(2)} + \xi_{a_2j_2}\xi_{a_3j_3}V^{a_1j_1\ a_2j_2\ a_3j_3}_{(3)} + \ldots \tag{3.33}$$

where $V^{a_1j_1\ldots a_nj_n}_{(n)}$ is assumed to be independent of ξ_{bk} and is antisymmetric under exchange of a_kj_k and a_lj_l indices. Then if one focuses on terms in $(Q+\hat{Q})(\xi_{aj}V^{aj})$ which are proportional to $(\lambda-\hat{\lambda})_{aj}$ and have no ξ_{aj} dependence, (3.31) implies that

$$(\lambda-\hat{\lambda})_{a_1j_1}V^{a_1j_1}_{(1)} = 0. \tag{3.34}$$

Furthermore, since $V^{a_1j_1}$ can only depend on $(\lambda-\hat{\lambda})_{aj}$ in the ghost-number zero combinations of the Lorentz currents $\lambda\gamma^{cd}w$ and $\hat{\lambda}\gamma^{cd}\hat{w}$, it is not difficult to show that (3.34) implies that $V^{a_1j_1}_{(1)} = 0$.

One can then focus on terms in $(Q+\hat{Q})(\xi_{aj}V^{aj})$ which are proportional to $(\lambda-\hat{\lambda})_{aj}$ and are linear in ξ_{aj}, and use a similar argument to prove that $V^{a_1j_1\ a_2j_2}_{(2)} = 0$. Continuing to higher powers in ξ_{aj}, one proves that $V^{aj}=0$ and therefore $S_0 = S_1$ in (3.30).

So it has been proven that T-duality invariance of the κ gauge-fixed Green-Schwarz action implies that the pure spinor version of the action is also invariant under T-duality.

4. Amplitudes and Wilson loops

4.1 Generalities on the amplitudes

In order to describe the external Yang Mills states it is convenient to use an on-shell superspace formalism where the superfields $\Phi(x,\theta)$ depend only on the eight chiral superspace variables θ^{ai}. We also find it convenient to write four dimensional on-shell momentum as

$$k_{a\dot{a}} = \pi_a\bar{\pi}_{\dot{a}} \tag{4.1}$$

which obeys $k^2=0$. An on-shell gluon supermultiplet is characterized by a momentum k and fermionic variables κ_i such that [39, 40]

$$\Phi_{k,\kappa}(x,\theta) = e^{ik\cdot x}e^{\pi_a\theta^{aj}\kappa_j} \tag{4.2}$$

Different components of the supermultiplet correspond to different terms in the κ expansion. The + helicity gluons correspond to the κ^0 terms and the − helicity gluons correspond to the κ^4 component.

The corresponding vertex operators in string theory have the form

$$V_{\pi,\bar{\pi},\kappa} = e^{i\pi_a\bar{\pi}_{\dot{a}}x^{a\dot{a}}} e^{\pi_b\theta^{bj}\kappa_j}\hat{V} \tag{4.3}$$

where $\hat{V}$ can contain only derivatives of θ and x. Of course, in addition it could contain other variables, such as $\bar{\theta}$, with or without derivatives. Thus the whole dependence on the x and θ zero modes of the vertex operators comes from the prefactor in (4.3) .

As we remarked above, before doing T-duality we should integrate out the zero modes of $x^{a\dot{a}}$ and θ^{aj}. This implies that the amplitude contains a factor

$$\mathcal{A} = \delta^4\left(\sum_{l=1}^{n} k^l_{a\dot{a}}\right)\ \delta^8\left(\sum_{l=1}^{n} \pi^l_a\kappa^l_i\right)\ \widetilde{\mathcal{A}} \tag{4.4}$$

We can extract physical amplitudes for individual polarization states from (4.4) by integrating over κ^l. Thus, if we simply integrate over κ^l we would be picking out the $(\kappa^l)^4$ term which is the minus helicity gluons. If we multiply by $(\kappa^l)^4$ and then integrate, then the lth particle corresponds to a + helicity gluon. This is equivalent to setting $\kappa^l = 0$ in (4.4) . The presence of the fermionic delta function in (4.4) implies that the all + amplitude and the almost all + and one − amplitude vanish. The first non-vanishing case is the MHV amplitude with mostly + and two − helicity gluons. For MHV amplitudes we do not need any further κ dependence in $\widetilde{\mathcal{A}}$, but amplitudes with more − helicities will require that we know the dependence of $\widetilde{\mathcal{A}}$ on κ. (A prescription for computing $\widetilde{\mathcal{A}}|_{\kappa=0}$ at tree level in string theory in flat space is given in appendix A.)

We introduce an infrared regularization as follows. We imagine starting from a $\mathrm{U}(N+k)$ theory. We consider a vacuum breaking the symmetry to $\mathrm{U}(N)\times\mathrm{U}(k)$ by giving a scalar field a vacuum expectation values μ_{IR} which will play the role of an infrared cutoff. When we take the 't Hooft limit we keep k fixed, so that the low energy $\mathrm{U}(k)$ theory becomes free. We then scatter n gluons of the $\mathrm{U}(k)$ theory. We are interested in the regime where all the kinematic invariants are much larger than the infrared scale, $s_{ij} \gg \mu^2_{\mathrm{IR}}$. On the strong coupling side, this infrared regularization corresponds to introducing k D3 branes in $AdS_5\times S^5$. In terms of the AdS metric $ds^2 = \frac{dx^2+dy^2}{y^2}$ the branes are sitting at $y = 1/\mu_{\mathrm{IR}}$. See figure 2. It is conceptually simpler for our purposes to say that $k = n$ and that the n gluons are open strings that stretch among these n branes so that each portion of the boundary of the disk diagram corresponds to each of the n branes.

4.2 Amplitudes after T-duality

After T-duality we can compute the quantity $\widetilde{\mathcal{A}}$ in the T-dual theory. We explain below what the corresponding computation is. The T-dual computation of $\widetilde{\mathcal{A}}$ involves a number of D(-1) branes and each external state maps to an open string stretching between the D(-1) branes, see figure 2. All the $D(-1)$ branes are sitting at the same $\widetilde{y}$ position $\widetilde{y} = \mu_{\mathrm{IR}}$. We can see that the open strings are stretched by looking at the original worldsheet equation of motion for one of the R^4 bosonic directions near the insertion point of the vertex operator (4.3) . It has the schematic form

$$k_1\delta^2(z) + \partial[g_{11}\bar{\partial}x^1 + \cdots] + \bar{\partial}[g_{11}\partial x^1 + \cdots] = k_1\delta^2(z) + (\partial\bar{\partial} - \bar{\partial}\partial)\widetilde{x}_1 \tag{4.5}$$

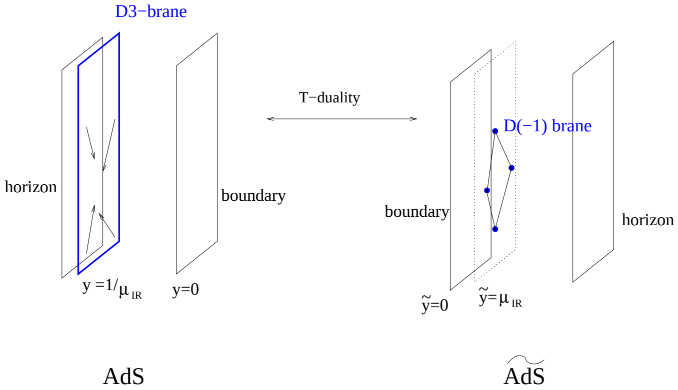

Figure 2: The amplitude computation in the original theory involves the scattering of open strings on n D3 branes living in AdS_5. Under T-duality this maps to a different computation in the T-dual AdS space. The T-dual computation involves strings stretching between n D(-1) branes. The D(-1) branes are positioned so that the open strings between them are massless. We are computing the interaction amplitude between these states in string theory which comes from a disk diagram.

where we have rewritten this equation in terms of the T-dual variable $\widetilde{x}^1$. Integrating this in an arc around the insertion of the vertex operator at the boundary we conclude that $\tilde{x}^1$ has winding given by k^1. In other words, the boundary condition for $\widetilde{x}^1$ changes from one side of the vertex operator to the other by an amount proportional to k_1 Of course this is the familiar statement that momentum is mapped into winding under T-duality. Let us now repeat this for the fermionic coordinates θ^{aj}. We find that the equation of motion is

$$\pi_a \kappa_i \delta^2(z) + \partial[C_{ai\,bj}\bar{\partial}\theta^{bj} + \cdots] - \bar{\partial}[C_{ai\,bj}\partial\theta^{bj} + \cdots] = \pi_a\kappa_i\delta^2(z) + (\partial\bar{\partial} - \bar{\partial}\partial)\widetilde{\theta}_{ai} \tag{4.6}$$

Thus we see that the T-dual fermionic coordinate $\widetilde{\theta}_{ai}$ has "winding" $\Delta\widetilde{\theta}_{ai} = \pi_a \kappa_i$ when we go across the vertex operator insertion. Thus we can assign to each D(-1) brane also a position in $\widetilde{\theta}$ which is consistent with these jumps. Notice that we will not integrate over the overall $\widetilde{\theta}$ fermion zero mode, so we are allowed to fix the position of one of these D(-1) branes arbitrarily. The same is true for the bosonic zero modes. One of the $D(-1)$ brane positions is fixed arbitrarily. We have n D(-1) branes, at specific separations given by the momenta and the fermionic coordinates κ_i of the external gluons. We have open strings stretching between them that are on-shell. Then we compute a disk diagram which is the tree level contribution to the interaction between these open strings. The whole computation is done in terms of the T-dual model, which is T-dual conformal invariant. The information about the polarizations of the gluons appears as the information about the particular open string state stretching between D(-1) branes that we are considering and it is encoded by the κ variables. We see that the theory, written in terms of the T-dual variables, has manifest dual superconformal symmetry, up to a small subtlety. If we consider the regularized amplitude, with a finite $\mu_{\rm IR}$, as in figure 2, then we map this to a configuration of D(-1) branes at the same position of the radial variable of the T-dual AdS space, $\tilde{y} = \mu_{\rm IR}$. However, a dual special conformal transformation will change their relative radial positions. In the limit that $\mu_{\rm IR} \to 0$, these positions are formally all at $\tilde{y} = 0$, which is the boundary of $\widetilde{AdS}$ space, and are unchanged by the conformal transformation.

However, the action diverges. Fortunately the structure of the divergences is known. After extracting the IR divergencies, one finds that the amplitude changes in a well defined way under such conformal transformation. The change is completely fixed by the structure of the IR divergencies. This was discussed in detail in [6] (see also [41] for a string perspective on the same issue.).

The bottom line is that the T-duality argument makes manifest the T-dual conformal symmetry and explains why it should be a symmetry of the amplitude. We have not been very explicit about the precise form of the vertex operators, but it seems clear that the symmetries are such that one should reproduce the structure described in [13] (and also [12]).

4.3 The amplitude and the Wilson loop

Let us now turn to the Wilson loop computation. The Wilson loop computation involves a string configuration very similar to the one that we get after performing the T-duality and taking $\mu_{\rm IR} \to 0$. One difference is that in the Wilson line computation there is no information about the polarization states of the gluons. This information arises in the T-dual computation as the polarization information for the strings stretching between $D(-1)$ branes. In order to obtain the Wilson loop, we need to "forget" about these polarization states and reduce the computation to one with fixed boundary conditions on the boundary of the string. For example we will put Dirichlet boundary conditions for the fermions and also for the AdS bosons. In the particular case of MHV amplitudes we expect that this change will simply produce a factor proportional to the tree level MHV amplitude. In other words, on the basis of the perturbative computations done in [3, 4, 9, 10], we expect that the relation is

$$\widetilde{\mathcal{A}}\Big|_{\kappa=0} = \frac{1}{\prod_{i=1}^{n}(\pi_i \pi_{i+1})} \langle W(k_1, \cdots, k_n) \rangle \tag{4.7}$$

up to IR and UV divergent terms. We do not have a rigorous justification for the origin of this prefactor on the string theory side. Of course, this factor accounts properly for the right helicity weights of the amplitude. It was also argued in [13] that it is dual superconformal covariant with weights one. So the only issue is whether one could get a residual superconformal invariant factor.

In lieu of a derivation, let us give some plausibility arguments. From the field theory side, as pointed out in [15] , when we put in this regularization we have an outer loop in the Feynmann diagram which consists of a massive supermultiplet. This particle mediates the interaction between the external $\mathrm{U}(k)$ particles and the $\mathrm{U}(N)$ internal particles. In the limit that we turn off the Yang Mills coupling of the U(N) theory, then we simply have a one loop diagram and we know that the result is equal to the MHV tree amplitude up to terms that capture the IR divergences [42] . From the string theory side, it is clear that the only difference between the amplitude and the Wilson loop computation lies in the detailed boundary conditions at the boundary of the worldsheet. (We explore the shape of the worldsheet near the boundary for a finite $\mu_{\rm IR}$ in appendix B.) Thus, it seems natural to expect that the worldsheet theory will contain some worldsheet excitations that are confined to the boundary of the worldsheet that would give rise to the prefactor in (4.7) . These could represent the massive particle we have in the field theory traveling around the loop. Then the difference between the amplitude and the Wilson loop would be whether we do or do not include these degrees of freedom localized at the boundary.

It is also quite plausible that we need to consider a Wilson loop with some insertions that take into account the polarization states of the particles. Studying the string theory in more detail one should be able to give a definite answer to these questions. It is also possible that one could understand this prefactor by computing precisely, as in [15] , the relation between MHV amplitudes and momentum space Wilson loops.

5. Dual conformal symmetry in the AdS sigma model

In this section we consider a bosonic sigma model with an AdS_{d+1} target space. Our goal is to get some insight on the connection between the dual conformal symmetry and integrability. The conserved currents associated to integrability in the original and T-dual model were studied in [43] and the flat connection of the T-dual model was written in terms of the variables of the original model. Our goal here is closely related. We will first relate the non-trivial dual conformal generators with the non-local currents that arise through integrability. We will also show the gauge equivalence of the flat connection of the original model and the one arising from the T-dual model.

As we mentioned above, writing the metric as

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \tag{5.1}$$

and performing T-duality in the x coordinates and an inversion of y

$$d\widetilde{x} = *\frac{dx}{y^2}\,, \qquad \widetilde{y} = \frac{1}{y}\,, \qquad dx = *\frac{d\widetilde{x}}{\widetilde{y}^2} \tag{5.2}$$

we can see that the equations of motion for $\widetilde{x}$ and $\widetilde{y}$ are the same as the ones we would obtain for a sigma model on the T-dual AdS space, or $\widetilde{AdS}_{d+1}$ space $ds^2 = \frac{d\tilde{x}^2 + d\tilde{y}^2}{\tilde{y}^2}$. The new AdS space has an SO$(2,d)$ symmetry group. Some of these symmetries are the same as the symmetries of the original model. For example, the dilatation symmetry D of the original model is related to the dilatation symmetry of the dual model, $D = -\widetilde{D}$. On the other hand, the special conformal symmetries of the dual model are not so obvious in the original model. We would like to understand what these symmetries are in the original model.

Let us consider first the simpler example of Euclidean AdS_2 or H_2. In this case we can write the special conformal generator of the dual model as

$$\widetilde{K} = \int d\sigma j_\tau^{\tilde{K}}(\sigma) = \int d\sigma \left[(\tilde{x}^2 - \tilde{y}^2)\frac{\partial_\tau \tilde{x}}{\tilde{y}^2} + 2\frac{\tilde{x}\partial_\tau \tilde{y}}{\tilde{y}} \right] \tag{5.3}$$

where τ and σ are the time and space coordinates on the worldsheet. We can now use (5.2) to replace the time derivatives of $\tilde{x}$ by sigma derivatives of x. We can then integrate these by parts. In this integral there are boundary terms and we assume that we can ignore these boundary terms (this will be true in the application we have in mind, where we will integrate on a closed contour and demand that x and $\tilde{x}$ are periodic). We are left with terms of the form $\tilde{x}\partial_\sigma\tilde{x}x$. We now write $\tilde{x}(\sigma) = \int^\sigma d\sigma'\partial_\sigma\tilde{x}$, and we replace the derivatives of $\tilde{x}$ by derivatives of x using (5.2) again. In the end we are left with an expression of the form

$$\tilde{K} = \int d\sigma \int^\sigma d\sigma' j^P_\tau(\sigma') j^D_\tau(\sigma) + \int d\sigma j^P_\sigma = P_2 \tag{5.4}$$

where $j^P \sim \frac{\partial x}{y^2}$, $j^D \sim \frac{xdx+ydy}{y^2}$ are the the translation and dilatation currents of the original model. Thus we see that the special conformal transformation in the dual model correspond to one of the non-local conserved charges. It is the second non-local conserved charge which is given by two integrals. Since the AdS model is integrable, we have an infinite set of non-local charges.

Thus, the conclusion is that the conformal symmetry of the dual model maps to the higher non-local charges of integrability. The same result is true in general AdS_{d+1} spaces. Thus, when we demand that a certain quantity is invariant under the dual conformal symmetry we are demanding that it is invariant under some of the non-local charges associated to integrability.

A simple way to think about these non-local charges is to construct a one parameter family of flat connections $\mathcal{C}(\lambda)$. This one parameter family can be used to write all the non-local conserved charges as we will review below. We can do the same for the dual model and construct $\widetilde{\mathcal{C}}(\lambda)$. We will then show that these two connections differ only by a gauge transformation, so that the total set of charges is the same on both sides.

5.1 Integrability and the flat connection

We work in AdS_{d+1}. This is described in terms of the coset manifold $\mathrm{SO}(2,d)/\mathrm{SO}(1,d)$ or G/H. We think of it as a right coset $g \sim gh$. The group G acts on the left and it corresponds to the global isometries of AdS. We will now construct the conserved currents for the model, following the discussion in [44] (see also [45]), with some minor changes in notation. We construct the left invariant (G invariant) currents

$$J = -g^{-1}dg \tag{5.5}$$

and we decompose them according to the decomposition of the Lie algebra $\mathcal{G} = \mathcal{H} + \mathcal{M}$, where $\mathcal{H}$ are the generators in the subgroup H and $\mathcal{M}$ are the rest. We then find

$$J = H + M \tag{5.6}$$

This transforms under $\mathcal{H}$ gauge transformations. The quantity

$$m = gMg^{-1} \tag{5.7}$$

is $\mathcal{H}$-invariant. The lagrangian can be written as $L \sim Tr[m_\alpha m^\alpha] \sim Tr[M_\alpha M^\alpha]$. For two quantities related as $x = gXg^{-1}$ we will use the lower case letter for the H invariant version and the upper case letter X for the G invariant one. We also note that

$$dx = gdXg^{-1} - j\wedge x - x\wedge j \tag{5.8}$$

where j corresponds to J. Since H is a subgroup we have $[\mathcal{H},\mathcal{H}] \subset \mathcal{H}$. Since we are performing a coset we also see that $[\mathcal{H},\mathcal{M}] \subset \mathcal{M}$. In our case we also have $[\mathcal{M},\mathcal{M}] \subset \mathcal{H}$.[4] From the definition (5.5) we know that $dJ = J \wedge J$. Decomposing J as in (5.6) and equating both sides we get

$$dH = H \wedge H + M \wedge M\,, \qquad\qquad dM = H \wedge M + M \wedge H \tag{5.9}$$

This then implies that

$$dm = -2m \wedge m \tag{5.10}$$

In addition we also have that m is proportional to the Noether current for the left G action. So $d * m = 0$. Thus we construct the flat connection as

$$\mathcal{C} = -2\sinh^2\frac{\lambda}{2}\; m + \sinh\lambda\; * m \tag{5.11}$$

where λ is an arbitrary complex parameter. This obeys $d\mathcal{C} + \mathcal{C} \wedge \mathcal{C} = 0$. One can then construct the holonomy

$$\Omega(\lambda) = P e^{\int \mathcal{C}(\lambda)} \tag{5.12}$$

Expanding this in powers of λ we get an infinite set of non-local conserved charges. The charge Q_n multiplying λ^n will contain a maximum of n integrals.

In the case of the cylinder we need to consider $Tr[\Omega^n]$ where Ω is the holonomy around the cylinder. These are then the conserved charges for a cylinder.

In the application to the amplitude we have a worldsheet which is a disk and thus we can form the holonomy around the origin of the disk. Since this can be smoothly deformed to the origin we conclude that the holonomy should be simply the identity matrix $\Omega = 1$. This is stating that the amplitude should be annihilated by all the charges, both the local and non-local charges.[5] We see that dual conformal symmetry corresponds to the statement that some particular charges annihilate the amplitude. Of course one needs to treat IR divergences carefully (see [6]), but this is the essence of the statement. It is natural to expect that demanding that all non-local symmetries annihilate the amplitude should determine the amplitude.

5.2 Relation between the flat connection in the original and the T-dual model

We now make a specific choice for the coset representative g as

$$g = e^{x.P} e^{\log y D} \tag{5.13}$$

where D is the dilatation operator and P_i are the momenta, $i = 1, \cdots, d$. We have $[D, P_i] = P_i$. We also have the special conformal generators K_j, $[D, K_j] = -K_j$,

[4] This can be seen as follows. Up to an irrelevant change in signature the coset is the same as $\mathrm{SO}(1, d+1)/\,\mathrm{SO}(d+1)$. Then $\mathcal{H}$ are all the rotation generators and $\mathcal{M}$ are all the boost generators. We know that the commutator of two boost generators is a rotation.

[5] This point of view was emphasized to us by A. Polyakov and A. Murugan.

$[K_i, P_j] = 2\delta_{ij}D$+rotation . Note that a combination of P and K, $\frac{1}{2}(P+K)$ is in $H = \mathrm{SO}(1,d)$ while the other combination is not. We have that

$$\begin{aligned}
J &= -\left[\frac{dy}{y}D + \frac{dx^i}{y}P_i\right] = -\left[\frac{dy}{y}D + \frac{dx^i}{y}\frac{1}{2}(P_i - K_i)\right] - \frac{dx^i}{y}\frac{1}{2}(P_i + K_i) \\
M &= -\left[\frac{dy}{y}D + \frac{dx^i}{y}\frac{1}{2}(P_i - K_i)\right] \\
H &= -\frac{dx^i}{y}\frac{1}{2}(P_i + K_i)
\end{aligned} \tag{5.14}$$

We can now construct the flat connection as in (5.11) . It is now convenient to do a gauge transformation of $\mathcal{C} \to \mathcal{C}' = g^{-1}\mathcal{C}g + g^{-1}dg$, where g is given in (5.13) . We then get

$$\begin{aligned}
\mathcal{C}' &= -2\sinh^2\frac{\lambda}{2}M + \sinh\lambda * M - (H+M) = -\cosh\lambda M + \sinh\lambda * M - H \\
\mathcal{C}' &= \left(\cosh\lambda\frac{dy}{y} - \sinh\lambda * \frac{dy}{y}\right)D + \qquad (5.15) \\
&\quad + \cosh\frac{\lambda}{2}\left(\cosh\frac{\lambda}{2}\frac{dx^i}{y} - \sinh\frac{\lambda}{2} * \frac{dx^i}{y}\right)P_i + \sinh\frac{\lambda}{2}\left(-\sinh\frac{\lambda}{2}\frac{dx^i}{y} + \cosh\frac{\lambda}{2} * \frac{dx^i}{y}\right)K_i
\end{aligned}$$

We can now construct a similar current in the T dual model, $\widetilde{\mathcal{C}}$, and then make a similar gauge transformation but in the T-dual model. We then get

$$\begin{aligned}
\widetilde{\mathcal{C}}' &= \left(\cosh\lambda\frac{d\tilde{y}}{\tilde{y}} - \sinh\lambda * \frac{d\tilde{y}}{\tilde{y}}\right)D + \qquad (5.16) \\
&\quad + \cosh\frac{\lambda}{2}\left(\cosh\frac{\lambda}{2}\frac{d\tilde{x}^i}{\tilde{y}} - \sinh\frac{\lambda}{2} * \frac{d\tilde{x}^i}{\tilde{y}}\right)P_i + \sinh\frac{\lambda}{2}\left(-\sinh\frac{\lambda}{2}\frac{d\tilde{x}^i}{\tilde{y}} + \cosh\frac{\lambda}{2} * \frac{d\tilde{x}^i}{\tilde{y}}\right)K_i
\end{aligned}$$

In principle we could have introduced another parameter $\tilde{\lambda}$ here. But, anticipating our result, we have set $\tilde{\lambda} = \lambda$. We can now express $\tilde{\mathcal{C}}'$ in terms of the original variables (x and y) via (5.2) . We then make an additional gauge transformation of $\mathcal{C}'$, this time by a constant group element, which maps $D \to -D$ and $P \leftrightarrow K$.

We then find

$$\begin{aligned}
\tilde{\mathcal{C}}'' &= \left(\cosh\lambda\frac{dy}{y} - \sinh\lambda * \frac{dy}{y}\right)D + \qquad (5.17) \\
&\quad + \cosh\frac{\lambda}{2}\left(\cosh\frac{\lambda}{2} * \frac{dx^i}{y} - \sinh\frac{\lambda}{2}\frac{dx^i}{y}\right)K_i + \sinh\frac{\lambda}{2}\left(-\sinh\frac{\lambda}{2} * \frac{dx^i}{y} + \cosh\frac{\lambda}{2}\frac{dx^i}{y}\right)P_i
\end{aligned}$$

We now note that the original flat connection $\mathcal{C}'$ can be related to $\tilde{\mathcal{C}}''$ via a gauge transformation by a constant group element

$$\mathcal{C}' = e^{-\mu D}\tilde{\mathcal{C}}'' e^{\mu D} \tag{5.18}$$

where μ is given by

$$e^{\mu} = \tanh\frac{\lambda}{2}, \qquad e^{-\mu D}Pe^{\mu D} = e^{-\mu}P, \qquad e^{-\mu D}Ke^{\mu D} = e^{\mu}K \tag{5.19}$$

We can see that expanding (5.18) in powers of λ one obtains a relation between non-local currents of different order. Notice that the gauge transformations we used prior to (5.18) were λ independent.

We have recently learnt that similar results, including a generalization to the full $AdS_5 \times S^5$ coset theory were obtained in [16] .

6. Conclusions

In this paper we have discussed the concept of "fermionic T-duality". We have shown that this is a symmetry of tree level string theory. At the level of the worldsheet we are performing the same steps as the ones we perform for a bosonic T-duality. We select a fermionic variable θ which has a shift symmetry. This corresponds to a supersymmetry that anticommutes to zero, $Q^2 = 0$. We then introduce the dual variable $\tilde{\theta}$ via equations that are similar to the ones we use for a bosonic T-duality. In target space this maps one supersymmetric background to another supersymmetric background. The RR fields and the dilaton are changed but the metric and the B field remain the same. In general, the reality conditions are not respected because we need a complex Killing spinor in order to have $Q^2 = 0$ for the corresponding supercharge. If we restrict to fermionic variables which are single-valued on the worldsheet, the T-duality will probably not extend to higher orders in string loops. On the other hand we expect it to be exact in α'. In fact, the change of the dilaton comes from a determinant that appears when we perform the change of variables in the path integral, as in the bosonic case [21, 22]. One example is the case of constant graviphoton background. This results from performing fermionic T-duality on a flat space background after adding a total derivative term to the action. Thus tree level string theory on a constant graviphoton background is the same as string theory on flat space. At higher string loop orders the two are different.

We have then applied this idea to the $AdS_5 \times S^5$ background. We performed four bosonic T-dualities along four translation symmetries of AdS_5 as well as eight fermionic T-dualities along the directions associated to the chiral Poincare supersymmetry generators Q_{ai} where a is a four dimensional chiral spinor index and i is a fundamental SU(4) R-symmetry index. After the dualities, the string theory comes back to itself. But the initial problem of computing scattering amplitudes translates into a problem involving a certain D(-1) brane configuration that is very similar to a Wilson loop configuration, with the D(-1) branes at the corners of the Wilson loop. The ordinary superconformal symmetry of the dual superstring theory is what was called "dual superconformal symmetry" of the original theory. Thus, this transformation makes this dual symmetry manifest. We have argued this for the classical Green Schwarz sigma model and then for the full quantum theory constructed using the pure spinor formalism. Our arguments amount to a change of variables in the path integral. We expect that there should be no anomalies associated to it. In particular, at one loop, we have checked that the Jacobian for this change of variables vanishes. Thus, we expect that the symmetry should be a full symmetry for any value of $\lambda = g^2 N$. In other words, we expect it to be exact in α'. This then explains the presence of the dual conformal symmetry found in weak coupling computations [12, 9, 10, 13]. It would also be nice to explain the emergence of this symmetry purely within the weak coupling theory. The four bosonic T-dualities are essentially a Fourier transform. This paper suggests that it would be productive to try to perform an additional transformation of the fermionic variables in order to be able to see the duality.

In the context of the simpler bosonic AdS sigma model we have also shown that the dual conformal symmetry amounts to some subset of the non-local charges associated to integrability. This has also been recently been done, including the extension to the full $AdS_5 \times S^5$ sigma model, in [16].

It has become clear that "dual superconformal symmetry" is very powerful in restricting the form of the amplitude. It even fixes the full amplitudes for four and five gluons [6] . Since this symmetry is simply a small part in the infinite set of conserved charges associated to integrability one would hope that all the higher charges can similarly be put to use in order to fully fix all amplitudes.

Fermionic T-dualities probably have many more applications that the one we used in this paper. In particular, since fermionic T-duality is a symmetry of supergravity, it seems that it might be possible to consider the continuous symmetry groups (e.g. E_7 [46]) that arise from toroidal compactifications and extend them to supergroups. If the current discussion of E_{10} and E_{11} models (see [47, 48] for recent papers) could be generalized to supergroup models, it might be possible to derive the d=10 and d=11 fermionic supergravity fields in the same manner as the bosonic supergravity fields have been derived in these models.

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A. MHV tree amplitudes in superstring theory

MHV tree amplitudes in flat space open superstring theory were studied in [19] using the RNS prescription. In this appendix, we propose a new prescription for computing MHV tree amplitudes in open superstring theory. Although our original motivation was to compute MHV superstring tree amplitudes in an $AdS_5 \times S^5$ background, up to now we have only been able to develop this prescription in a flat background. Nevertheless, this flat space prescription for computing superstring MHV tree amplitudes is simpler than previous prescriptions and has an interesting relationship with the self-dual N=2 string of [17, 18]. Such a relationship is not surprising since the self-dual N=2 string computes self-dual d=4 Yang-Mills amplitudes which have many features in common with MHV amplitudes [49 − 51].

A.1 MHV tree amplitudes in gauge theory

N-point tree-level MHV amplitudes have an extremely simple form when expressed in terms of spinor helicities. If the $d = 4$ light-like momentum $p_{a\dot{a}} = p_m \sigma^m_{a\dot{a}}$ of the r^{th} state is written as

$$p_r^{a\dot{a}} = \pi_r^a \bar{\pi}_r^{\dot{a}}, \tag{A.1}$$

the color-ordered N-point tree-level MHV amplitude with $N - 2$ self-dual gluons and 2 anti-self-dual gluons is

$$A = \frac{(\pi_J \pi_K)^4}{\prod_{r=1}^{N} (\pi_r \pi_{r+1})} \tag{A.2}$$

where J and K label the anti-self-dual gluons, $\pi_{N+1} \equiv \pi_1$, and the color factor $Tr(T^{a_1} \dots T^{a_N})$ has been suppressed. In (A.2), the self-dual gluon polarization is $\eta_r^{a\dot{a}} = \varepsilon_r^a \bar{\pi}_r^{\dot{a}}$ and the anti-self-dual gluon polarization is $\bar{\eta}_r^{a\dot{a}} = \pi_r^a \varepsilon_r^{\dot{a}}$ where ε_r^a and $\varepsilon_r^{\dot{a}}$ are normalized such that $\varepsilon_r^a \pi_{ra} = 1$ and $\varepsilon_r^{\dot{a}} \bar{\pi}_{r\dot{a}} = 1$.

The formula (A.2) can be easily extended to describe the scattering of any $\mathcal{N} = 4$ super-Yang-Mills fields by combining the $\mathcal{N}$=4 super-Yang-Mills fields into a scalar chiral superfield $\Phi(x, \theta)$. For an on shell gluon the field has a special form characterized by its momentum and some fermionic parameters κ^i determining its various components. We have

$$\Phi_{p,\kappa}(x, \theta) = e^{ip.x} e^{\pi_a \kappa_i \theta^{ai}} \tag{A.3}$$

Expanding in powers of κ we obtain the various components of the superfield. We can think about the amplitude as a function of $\pi^a, \bar{\pi}^{\dot{a}}, \kappa_i$ for each gluon. By looking at the $(\kappa^r)^4$ term we extract the amplitude for the negative helicity gluon, while the $(\kappa^r)^0$ term corresponds to the positive helicity gluon.

The amplitude will contain an integral over θ that will translate into an overall factor of the form

$$\mathcal{A}(\pi^r, \bar{\pi}^r, \kappa^r) = \delta^4\Big(\sum_r p_r\Big) \delta^8\Big(\sum_r \pi_r^a \kappa_i^r\Big) \widetilde{\mathcal{A}} \tag{A.4}$$

The amplitude $\widetilde{\mathcal{A}}$ could have additional κ dependence. However, the κ independent part of $\widetilde{\mathcal{A}}$ is the MHV amplitude, up to the prefactor in (A.4) .

In field theory we find that this MHV part is given by

$$\widetilde{\mathcal{A}} = \frac{1}{\prod_{r=1}^{N} (\pi_r \pi_{r+1})} \tag{A.5}$$

The numerator factor in (A.2) comes from considering the δ function in (A.4) and integrating over four of the κ's for each of the negative helicity gluons.

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A.2 MHV tree amplitude in open superstring theory

The arguments leading to (A.4) were completely kinematical and also hold for open superstring theory. Namely, they also hold if we consider open string scattering for massless open strings on a D3 brane, even in the case that the scattering occurs at energies higher than the string scale. In that case the MHV amplitude will not be given by (A.5) and will contain dependence on α' . In this subsection we propose a way to compute $\widetilde{\mathcal{A}}$ in flat space open superstring theory.

We conjecture that the MHV superstring amplitude is given by

$$\begin{aligned}\widetilde{\mathcal{A}}(\pi_r, \bar{\pi}_r) = (\pi_1\pi_2)^{-1}(\pi_2\pi_3)^{-1}(\pi_3\pi_1)^{-1} \times \\ \langle V_1(z_1)V_2(z_2)V_3(z_3)\int_{z_3}^{z_1} dz_4 U_4(z_4)\ldots\int_{z_{N-1}}^{z_1} dz_N U_N(z_N)\rangle\end{aligned} \tag{A.6}$$

where $V_r(z_r) = e^{i\pi_r\bar{\pi}_r x(z_r)}$ and

$$U_r(z_r) = (\varepsilon_r^a \bar{\pi}_r^{\dot{a}} \partial x_{a\dot{a}} + \psi_{\dot{a}}\bar{\psi}_{\dot{b}}\bar{\pi}_r^{\dot{a}}\bar{\pi}_r^{\dot{b}})e^{i\pi_r\bar{\pi}_r x(z_r)}. \tag{A.7}$$

The correlation function in (A.6) is defined in the usual manner where $x_{a\dot{a}}(z)$ satisfies the OPE $x_{a\dot{a}}(y)x_{b\dot{b}}(z) \to -\alpha'\epsilon_{ab}\epsilon_{\dot{a}\dot{b}}(\log|y-z| + \log|y-\bar{z}|)$ and $(\psi_{\dot{a}}, \bar{\psi}_{\dot{b}})$ are fermions of conformal weight $(\frac{1}{2}, 0)$ satisfying the OPE $\psi_{\dot{a}}(y)\bar{\psi}_{\dot{b}}(z) \to \alpha'\epsilon_{\dot{a}\dot{b}}(y-z)^{-1}$. If $(\psi_{\dot{a}}, \bar{\psi}_{\dot{b}})$ are relabeled as $\psi_{a\dot{a}}$, the vertex operator $\int dz\, \mathrm{U}(z)$ of (A.7) is the standard RNS vertex operator for a self-dual gluon. Note that $U_r(z_r)$ changes by a total derivative under the gauge transformation $\delta\varepsilon_r^a = c\pi_r^a$ for any constant c, so with the normalization $\varepsilon_r^a\pi_{ra} = 1$, the amplitude is independent of ε_r^a.

The novelty of (A.6) is that the computation of the superstring MHV tree amplitude is manifestly invariant under $N = 4$ $d = 4$ spacetime supersymmetry. Although one can of course compute superstring MHV tree amplitudes using either the RNS or pure spinor formalism, computations using these formalisms are more complicated and contain many more fields. Note that (A.6) depends only on $\pi_r\bar{\pi}_r$ but not on κ. This is because we are concentrating on MHV amplitudes and we are only giving a prescription for computing MHV amplitudes. We are not saying how to compute non-MHV amplitudes, which should contain some κ dependence.

We will not attempt to derive (A.6) from a superstring formalism, however, there is an interesting relation to the open self-dual string with N=2 worldsheet supersymmetry [17, 18]. This open string theory has a single physical state in its spectrum corresponding to a self-dual Yang-Mills gluon (in signature $d = (2,2)$). The worldsheet matter variables in the self-dual string consists of $(x^{a\dot{a}}, \psi_{\dot{a}}, \bar{\psi}_{\dot{a}})$ with $\hat{c} = 2$ $N = 2$ superconformal generators

$$T = \frac{1}{2}\partial x^{a\dot{a}}\partial x_{a\dot{a}} + \frac{1}{2}(\psi^{\dot{a}}\partial\bar{\psi}_{\dot{a}} + \bar{\psi}^{\dot{a}}\partial\psi_{\dot{a}}), \quad G^+ = \psi_{\dot{a}}\partial x^{+\dot{a}}, \quad G^- = \bar{\psi}_{\dot{a}}\partial x^{-\dot{a}}, \quad J = \psi^{\dot{a}}\bar{\psi}_{\dot{a}}, \tag{A.8}$$

and worldsheet action

$$\frac{1}{\alpha'}\int d^2z\left[\frac{1}{2}\partial x^{a\dot{a}}\bar{\partial}x_{a\dot{a}} + \bar{\psi}^{\dot{a}}\bar{\partial}\psi_{\dot{a}}\right]. \tag{A.9}$$

The physical self-dual Yang-Mills state is associated with the $N = 2$ superconformal primary field

$$V = \exp(ip_{a\dot{a}}x^{a\dot{a}}), \tag{A.10}$$

and the integrated vertex operator is

$$\int dzG^-G^+V = \int dz\pi^-(\bar{\pi}_{\dot{a}}\partial x^{+\dot{a}} + \pi^+(\psi_{\dot{a}}\bar{\pi}^{\dot{a}})(\bar{\psi}_{\dot{a}}\bar{\pi}^{\dot{a}}))e^{ipx} \tag{A.11}$$

where $p_{a\dot{a}} = \pi_a\bar{\pi}_{\dot{a}}$. So if one chooses the gauge $\epsilon^+ = 0$ and $\epsilon^- = \frac{1}{\pi^+}$, (A.11) is equal to $\pi^+\pi^- \int dz\, \mathrm{U}(z)$ where $\mathrm{U}(z)$ is defined in (A.7).

Using the "topological" rules of [52] for computing self-dual open string amplitudes, the N-point tree amplitude prescription is

$$\mathcal{A}_{\mathcal{N}=2} = \langle (G^+V(z_1))(G^+V(z_2))V(z_3)\prod_{r=4}^{N}\int dz_rU_r(z_r)\rangle \tag{A.12}$$

where the N=2 superconformal generators of (A.8) have been twisted so that $\psi_{\dot{a}}$ carries zero conformal weight and the zero-mode measure factor is $\langle\psi_{\dot{a}}\psi^{\dot{a}}\rangle = 1$. As shown in [52], these N-point amplitudes vanish when $N > 3$ as is expected for self-dual Yang-Mills tree amplitudes.

The N-point tree amplitude prescription proposed here is slightly different from (A.12) and is

$$\widetilde{\mathcal{A}} = (\pi_1\pi_2)^{-1}(\pi_2\pi_3)^{-1}(\pi_3\pi_1)^{-1}\langle V(z_1)\ V(z_2)\ V(z_3)\ \prod_{r=4}^{N}\int dz_rU_r(z_r)\rangle \tag{A.13}$$

where the N=2 superconformal generators are untwisted. As shown below, this new prescription is non-vanishing for $N > 3$ and reproduces the gauge theory result of (A.5) in the limit when $\alpha' \to 0$.

This suggests that there should be a superstring formalism which combines the worldsheet variables of the self-dual string with another sector containing θ^{aj} worldsheet variables. One possibility for such a formalism is the self-dual super-Yang-Mills string theory constructed in [53], which is related to the Green-Schwarz self-dual string of [54]. It would be very interesting if one could use this formalism to construct a prescription with manifest $N = 4$ $d = 4$ supersymmetry which reproduces superstring non-MHV tree amplitudes.

A.3 $\alpha' \to 0$ limit of superstring tree amplitude

The first step in checking the validity of (A.6) is to show that it reproduces the gauge theory amplitude of (A.5) in the limit when $\alpha' \to 0$. To evaluate (A.6), it is convenient to use N=2 notation and express the vertex operator of (A.7) as

$$U_r(z) = \int d\chi_r \int d\bar{\chi}_r\ \exp[\pi_r^a\bar{\pi}_r^{\dot{a}}x_{a\dot{a}}(z) + \chi_r(\bar{\pi}_r\psi(z)) + \bar{\chi}_r(\bar{\pi}_r\bar{\psi}(z)) + \chi_r\bar{\chi}_r\epsilon_r^a\bar{\pi}_r^{\dot{a}}\partial x_{a\dot{a}}(z)] \tag{A.14}$$

where χ_r and $\bar{\chi}_r$ are Grassmann parameters which are introduced simply as a technical trick. Using the free-field OPE's implied by (A.9), one finds

$$\begin{aligned}
&\langle V(z_1)\; V(z_2)\; V(z_3)\; \prod_{r=4}^{N} \int dz_r U_r(z_r)\rangle \qquad\qquad (A.15)\\
&\quad = \prod_{r=4}^{N} \int dz_r d\chi_r d\bar{\chi}_r \prod_{r,s} |z_r - z_s|^{\alpha' p_r p_s} \times \\
&\qquad\qquad \times \exp\left[\alpha' \frac{\bar{\pi}_r \bar{\pi}_s}{z_r - z_s} (\chi_r \bar{\chi}_r (\epsilon_r \pi_s) + \chi_s \bar{\chi}_s (\epsilon_s \pi_r) + \chi_r \bar{\chi}_s + \chi_s \bar{\chi}_r)\right]
\end{aligned}$$

where we have chosen a gauge for the ϵ_r's such that $\epsilon_r^a \epsilon_{sa} = 0$ for all r and s. Note that

$$\begin{aligned}
&\exp\left[\alpha' \frac{\bar{\pi}_r \bar{\pi}_s}{z_r - z_s} (\chi_r \bar{\chi}_r (\epsilon_r \pi_s) + \chi_s \bar{\chi}_s (\epsilon_s \pi_r) + \chi_r \bar{\chi}_s + \chi_s \bar{\chi}_r)\right] \qquad (A.16)\\
&\quad = 1 + \alpha' \frac{\bar{\pi}_r \bar{\pi}_s}{z_r - z_s} (\chi_r \bar{\chi}_r (\epsilon_r \pi_s) + \chi_s \bar{\chi}_s (\epsilon_s \pi_r) + \chi_r \bar{\chi}_s + \chi_s \bar{\chi}_r)]
\end{aligned}$$

and has no double poles when $z_r - z_s \to 0$.

Since each term in the exponential of (A.16) is proportional to α', these terms can only contribute in the limit $\alpha' \to 0$ if there appear factors of $\frac{1}{\alpha'}$ coming from the integration over z_r. Such factors of $\frac{1}{\alpha'}$ can arise from contact terms when $z_{r-1} \to z_r$ since

$$\int_{z_{r-1}}^{z_{r-1}+\Delta} dz_r\; |z_r - z_{r-1}|^{\alpha' p_r p_{r-1} - 1} = (\alpha' p_r p_{r-1})^{-1} \qquad (A.17)$$

for arbitrarily small Δ. So the terms in (A.16) can only contribute if they are proportional to $(z_r - z_{r-1})^{-1}$, i.e. if they involve neighboring vertex operators.

After integrating over $\prod_{r=4}^{N} d\chi_r d\bar{\chi}_r$ and taking the limit $\alpha' \to 0$, one finds that (A.15) is equal to

$$\begin{aligned}
&\lim_{\alpha' \to 0} \prod_{r=4}^{N} \int dz_r |z_r - z_{r-1}|^{\alpha' p_r p_{r-1}} |z_1 - z_N|^{\alpha' p_1 p_N} \qquad\qquad (A.18)\\
&\sum_{s=0}^{N-3} \left[\prod_{t=4}^{N-s} \frac{\alpha' (\bar{\pi}_t \bar{\pi}_{t-1})(\epsilon_t \pi_{t-1})}{z_t - z_{t-1}} \prod_{t=N-s+1}^{N} \frac{\alpha' (\bar{\pi}_t \bar{\pi}_{t+1})(\epsilon_t \pi_{t+1})}{z_t - z_{t+1}} \right]\\
&\quad = \sum_{s=0}^{N-3} \left[\prod_{t=4}^{N-s} \frac{(\bar{\pi}_t \bar{\pi}_{t-1})(\epsilon_t \pi_{t-1})}{p_t p_{t-1}} \prod_{t=N-s+1}^{N} \frac{(\bar{\pi}_t \bar{\pi}_{t+1})(\epsilon_t \pi_{t+1})}{-p_t p_{t+1}} \right]\\
&\quad = \sum_{s=0}^{N-3} \left[\prod_{t=4}^{N-s} \frac{(\epsilon_t \pi_{t-1})}{(\pi_t \pi_{t-1})} \prod_{t=N-s+1}^{N} \frac{(\epsilon_t \pi_{t+1})}{(\pi_{t+1} \pi_t)} \right]\\
&\quad = (\pi_3 \pi_1) \prod_{r=3}^{N} (\pi_r \pi_{r+1})^{-1}. \qquad\qquad (A.19)
\end{aligned}$$

Finally, multiplying (A.19) by the first line of (A.6), one reproduces the MHV gauge theory amplitude of (A.5).

A.4 Comparison with four-point and five-point gluon amplitudes

A second check of the conjecture of (A.6) is that it correctly reproduces the four-point and five-point gluon scattering when all polarizations and momenta are four-dimensional.

For four-point scattering, the correlation function in (A.6) contributes

$$\begin{aligned}\int_{z_3}^{z_1} dz_4 \sum_{s=1}^{3} \frac{(\varepsilon_4\pi_s)(\bar\pi_4\bar\pi_s)}{z_4 - z_s} \prod_{r,s} |z_r - z_s|^{\alpha'(\pi_r\pi_s)(\bar\pi_r\bar\pi_s)} \\ = \int_0^1 dz_4 \frac{(\pi_3\pi_1)(\bar\pi_4\bar\pi_1)}{(z_4-1)(\pi_3\pi_4)} \prod_{r,s} |z_r - z_s|^{\alpha'(\pi_r\pi_s)(\bar\pi_r\bar\pi_s)}\end{aligned} \tag{A.20}$$

where ε_4^a has been gauged to $\varepsilon_4^a = \pi_3^a(\pi_3\pi_4)^{-1}$ and (z_1, z_2, z_3) have been set to $(1, \infty, 0)$. Multiplying by the first line of (A.6), one obtains the amplitude

$$\begin{aligned}\widetilde{\mathcal{A}} &= \frac{(\bar\pi_1\bar\pi_4)}{(\pi_3\pi_4)(\pi_2\pi_3)(\pi_1\pi_2)} \frac{\Gamma(-\alpha' s+1)\Gamma(-\alpha' t)}{\Gamma(\alpha' u+1)} \\ \widetilde{\mathcal{A}} &= \prod_{r=1}^{N=4} (\pi_r\pi_{r+1})^{-1} \frac{\Gamma(-\alpha' s+1)\Gamma(-\alpha' t+1)}{\Gamma(\alpha' u+1)}\end{aligned} \tag{A.21}$$

which is the correct open superstring four-point amplitude.

For five-point scattering, the correlation function in (A.6) contributes

$$\begin{aligned}&\int_{z_3}^{z_1} dz_4 \int_{z_4}^{z_1} dz_5 \prod_{r,s} |z_r - z_s|^{\alpha'(\pi_r\pi_s)(\bar\pi_r\bar\pi_s)} \\ &\quad \left[\sum_{r\neq 4} \frac{(\varepsilon_4\pi_s)(\bar\pi_4\bar\pi_s)}{z_4 - z_s} \sum_{s\neq 5} \frac{(\varepsilon_5\pi_s)(\bar\pi_5\bar\pi_s)}{z_5 - z_s} - \frac{(\bar\pi_4\bar\pi_5)^2}{(z_4-z_5)^2} - \frac{(\varepsilon_4\varepsilon_5)(\bar\pi_4\bar\pi_5)}{\alpha'(z_4-z_5)^2}\right] \\ &= \int_1^\infty dz_4 \int_{z_4}^\infty dz_5 \prod_{r,s} |z_r - z_s|^{\alpha'(\pi_r\pi_s)(\bar\pi_r\bar\pi_s)} \\ &\quad \left[\left(\frac{(\bar\pi_4\bar\pi_5)(\pi_3\pi_5)}{(\pi_3\pi_4)(z_4-z_5)} + \frac{(\bar\pi_4\bar\pi_2)(\pi_3\pi_2)}{(\pi_3\pi_4)(z_4-z_2)}\right)\left(\frac{(\bar\pi_5\bar\pi_4)(\pi_3\pi_4)}{(\pi_3\pi_5)(z_5-z_4)} + \frac{(\bar\pi_5\bar\pi_2)(\pi_3\pi_2)}{(\pi_3\pi_5)(z_5-z_2)}\right) - \frac{(\bar\pi_5\bar\pi_4)^2}{(z_4-z_5)^2}\right] \\ &= \int_1^\infty dz_4 \int_{z_4}^\infty dz_5 \prod_{r,s} |z_r - z_s|^{\alpha'(\pi_r\pi_s)(\bar\pi_r\bar\pi_s)} \\ &\quad \frac{(\pi_3\pi_2)}{(\pi_3\pi_4)(\pi_3\pi_5)} \left[\frac{(\bar\pi_4\bar\pi_5)(\bar\pi_5\bar\pi_2)(\pi_3\pi_5)}{(z_4-z_5)(z_5-z_2)} + \frac{(\bar\pi_4\bar\pi_2)(\bar\pi_5\bar\pi_4)(\pi_3\pi_4)}{(z_4-z_2)(z_5-z_4)} + \frac{(\bar\pi_4\bar\pi_2)(\bar\pi_5\bar\pi_2)(\pi_3\pi_2)}{(z_4-z_2)(z_5-z_2)}\right]\end{aligned} \tag{A.22}$$

where ε_4^a and ε_5^a have been gauged to $\varepsilon_4^a = \pi_3^a(\pi_3\pi_4)^{-1}$ and $\varepsilon_5^a = \pi_3^a(\pi_3\pi_5)^{-1}$, and (z_1, z_2, z_3) have been set to $(-\infty, 0, 1)$.

Defining $z_4 = x^{-1}$ and $z_5 = (xy)^{-1}$ as in [19], the integral

$$\begin{aligned}&\int_1^\infty dz_4 \int_{z_4}^\infty dz_5 \prod_{r,s} |z_r - z_s|^{\alpha' s_{rs}} \\ &\quad \left[\frac{A}{(z_4-z_5)(z_5-z_2)} + \frac{B}{(z_4-z_2)(z_5-z_4)} + \frac{C}{(z_4-z_2)(z_5-z_2)}\right] \\ &= A\frac{s_{35}f_2 - s_{15}f_1}{s_{45}} + B\left(f_1 - \frac{s_{35}f_2 - s_{15}f_1}{s_{45}}\right) + Cf_1\end{aligned} \tag{A.23}$$

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where $s_{rs} = (\pi_r \pi_s)(\bar{\pi}_r \bar{\pi}_s)$,

$$\begin{aligned} f_1 &= \int_0^1 dx \int_0^1 dy x^{-1} y^{-1} \mathcal{I}(x,y), \\ f_2 &= \int_0^1 dx \int_0^1 dy (1-xy)^{-1} \mathcal{I}(x,y), \\ \mathcal{I}(x,y) &= x^{\alpha' s_{23}} y^{\alpha' s_{51}} (1-x)^{\alpha' s_{34}} (1-y)^{\alpha' s_{45}} (1-xy)^{\alpha'(s_{12}-s_{34}-s_{45})}. \end{aligned} \tag{A.24}$$

Plugging in

$$A = \frac{(\pi_3\pi_2)(\bar{\pi}_4\bar{\pi}_5)(\bar{\pi}_5\bar{\pi}_2)}{(\pi_3\pi_4)}, \quad B = \frac{(\pi_3\pi_2)(\bar{\pi}_4\bar{\pi}_2)(\bar{\pi}_5\bar{\pi}_4)}{(\pi_3\pi_5)}, \quad C = \frac{(\bar{\pi}_4\bar{\pi}_2)(\bar{\pi}_5\bar{\pi}_2)(\pi_3\pi_2)^2}{(\pi_3\pi_4)(\pi_3\pi_5)}, \tag{A.25}$$

using the identity $\sum_s (\pi_r\pi_s)(\bar{\pi}_s\bar{\pi}_t) = 0$ which follows from the momentum conservation of $\sum_s p_s = 0$, and multiplying by the factor in the first line of (A.6), one obtains

$$\widetilde{\mathcal{A}} = \frac{1}{\prod_{s=1}^{N=5}(\pi_s\pi_{s+1})}[s_{51}s_{23}f_1 + (\pi_5\pi_1)(\bar{\pi}_1\bar{\pi}_2)(\pi_2\pi_3)(\bar{\pi}_3\bar{\pi}_5)f_2] \tag{A.26}$$

which agrees with the five-point gluon amplitude of [19].

A.5 BRST operator

Since the form of the unintegrated operators V and integrated operators U look very different in (A.6), it is far from obvious that the superstring formula of (A.6) is invariant under cyclic permutations of the N states. In the following subsections, we will give an argument for this cyclic symmetry which involves picture-changing operators. However, these arguments are not rigorous and it would certainly be useful to better understand this point.

To argue that the prescription has cyclic symmetry, it is convenient to first define the nilpotent operator

$$Q = \int dz (\lambda^\alpha \bar{\psi}^{\dot{a}} \partial x_{\alpha\dot{a}} + e\bar{\psi}_{\dot{a}}\bar{\psi}^{\dot{a}} + f\lambda^\alpha\partial\lambda_\alpha) \tag{A.27}$$

where λ^α is a bosonic spinor of conformal weight $(-\frac{1}{2}, 0)$ and e and f are conjugate fermions of conformal weight $(0,0)$ and $(1,0)$ which satisfy the OPE $e(y)f(z) \to \alpha'(y-z)^{-1}$. This nilpotent operator will be called a BRST operator for reasons that will become clear shortly.

Using the free-field OPE's of $(x^{a\dot{a}}, \psi_{\dot{a}}, \bar{\psi}_{\dot{a}})$, one can verify that $QU_r = \partial S_r$ where

$$S_r = (\lambda\varepsilon_r)(\bar{\pi}_r\bar{\psi})e^{i\pi_r\bar{\pi}_r x} \tag{A.28}$$

satisfies $QS_r = 0$. Furthermore, under $\delta\varepsilon_r^a = c\pi_r^a$, $\delta S_r = Q\Omega_r$ where $\Omega_r = e^{i\pi_r\bar{\pi}_r x}$.

Naively, one would compute BRST-invariant tree amplitudes by evaluating the correlation function of 3 unintegrated vertex operators S_r and $N-3$ integrated vertex operators $\int dz U_r$. However, this would give an inconsistent result for two reasons. Firstly, the 3 unintegrated vertex operators would contribute three factors of $\bar{\psi}$, whereas $\bar{\psi}^{\dot{a}}$ has no zero modes since it has conformal weight $(\frac{1}{2}, 0)$. And secondly, the $-\frac{1}{2}$ conformal weight of λ^α

implies that it has bosonic zero modes on a disk. As will be explained below, λ^α has 3 bosonic zero modes on a disk and integration over these non-compact bosonic zero modes would give a factor of $(\infty)^3$ if the correlation function were defined using the above vertex operators.

To obtain the appropriate zero mode factors, one needs to replace the vertex operators S_r of (A.28) with vertex operators in a lower "picture". These picture-lowered vertex operators W_r will be defined as

$$W_r = (\lambda\varepsilon_r)\delta(\lambda\pi_r)e^{i\pi_r\bar{\pi}_r x} \tag{A.29}$$

where $\delta(\lambda\pi_r)$ denotes a delta-function which constrains one of the three zero modes of λ^α. It is easy to check that $QW_r = 0$ and that W_r is invariant under the gauge transformation $\delta\varepsilon_r = c\pi_r$.

To understand the relation between W_r of (A.29) and S_r of (A.28), note that $S_r = Q\Sigma_r$ where

$$\Sigma_r = \frac{(\lambda\varepsilon_r)}{(\lambda\pi_r)}e^{i\pi_r\bar{\pi}_r x}. \tag{A.30}$$

So if Σ_r were a well-defined state, S_r would be BRST-trivial. This situation has an analog in the RNS formalism since any BRST-closed state V_{RNS} can be written as $V_{\mathrm{RNS}} = Q_{\mathrm{RNS}}\Sigma_{\mathrm{RNS}}$ where $\Sigma_{\mathrm{RNS}} = c\xi\partial\xi e^{-2\phi}V_{\mathrm{RNS}}$ and $(\eta e^\phi, \partial\xi e^{-\phi})$ is the bosonized version of the (γ,β) RNS ghosts. In this case, Σ_{RNS} is not a well-defined state since it depends on the ξ zero mode, i.e. $\eta_0 \equiv \int dz\eta$ does not annihilate Σ_{RNS}. However, $W_{\mathrm{RNS}} = \eta_0\Sigma_{\mathrm{RNS}} = c\partial\xi e^{-2\phi}V_{\mathrm{RNS}}$ is a well-defined state and defines the picture-lowered version of the vertex operator. Note that $W_{\mathrm{RNS}} = YV_{\mathrm{RNS}}$ where $Y = c\partial\xi e^{-2\phi}$ is the picture-lowering operator satisfying $YX = 1$, and $X = \{Q_{\mathrm{RNS}}, \xi\}$ is the picture-raising operator. [55]

To mimic this situation in RNS, suppose that $\lambda_a\pi_r^a$ is bosonized as

$$(\lambda\pi) = \eta e^\phi. \tag{A.31}$$

This means that Σ_r of (A.30) can be expressed as

$$\Sigma_r = \xi e^{-\phi}(\lambda\varepsilon_r)e^{i\pi_r\bar{\pi}_r x}. \tag{A.32}$$

Defining the picture-lowered vertex operator as $W_r = \eta_0\Sigma_r = e^{-\phi}(\lambda\varepsilon_r)e^{i\pi_r\bar{\pi}_r x}$, one obtains the operator of (A.29) if $e^{-\phi}$ is identified as $\delta(\lambda\pi)$. This identification is very natural and is analogous to the identification of $e^{-\phi} = \delta(\gamma)$ in the RNS formalism [56].

A.6 Cyclic symmetry

In this subsection, it will be shown that the amplitude of (A.6) can be expressed as

$$\widetilde{\mathcal{A}} = \langle W_1(z_1)W_2(z_2)W_3(z_3)\int_{z_3}^{z_1} dz_4U_4(z_4)\ldots\int_{z_{N-1}}^{z_1} dz_NU_N(z_N)\rangle \tag{A.33}$$

where the vertex operators W_r and U_r are defined in (A.29) and (A.7) and the correlation function in (A.33) includes functional integration over the λ^α zero modes. Since the unintegrated vertex operators W_r and the integrated vertex operators U_r are related by picture-changing operators, one expects to be able to use the usual picture-changing arguments of [55] to prove that (A.33) is invariant under cyclic symmetry.

Since λ^a has conformal weight $-\frac{1}{2}$, each component of λ^a has two zero modes on a disk, i.e. $\lambda^a(z) = A^a + zB^a$ where A^a and B^a are zero modes. However, the BRST operator and all vertex operators are invariant under the rescaling

$$\lambda^a \to C\lambda^a, \quad \bar{\psi}^{\dot{a}} \to C^{-1}\bar{\psi}^{\dot{a}}, \quad \psi^{\dot{a}} \to C\psi^{\dot{a}}, \quad e \to C^2 e, \quad f \to C^{-2} f, \tag{A.34}$$

so one of the four zero modes can be gauged away. The integral over the remaining three zero modes can be easily performed using the result that

$$\int dA^1 dA^2 dB^1 \lambda^a(z_1)\lambda^b(z_2)\lambda^c(z_3)\delta(\lambda(z_1)\pi_1)\delta(\lambda(z_2)\pi_2)\delta(\lambda(z_3)\pi_3) \\ = \pi_1^a\pi_2^b\pi_3^c(\pi_1\pi_2)^{-1}(\pi_2\pi_3)^{-1}(\pi_3\pi_1)^{-1}. \tag{A.35}$$

After plugging (A.35) into (A.33), one easily verifies that (A.33) reproduces the amplitude prescription of (A.6). So assuming that the picture-changing manipulations of [55] can be applied to this situation, (A.6) has been shown to be invariant under cyclic permutations of the vertex operators.

B. Cusp solution for the brane regularization

In this appendix we find the classical solution describing a string ending on a cusp that is sitting at $z = \epsilon$, near the boundary of AdS space. This is a generalization of the solution in [57] which describes the case $\epsilon = 0$.

We focus on an AdS_3 subspace of AdS_5 which is parametrized by $x^\pm$ and the radial coordinate z. We want to find the surface that ends on the cusp given by $x^+x^- = 0$ (only the part in the forward lightcone) and at $z = \epsilon$.

We assume boost invariance so that the solution depends only on one variable. Let us define variables so that $x^\pm = e^{\tau\pm\sigma}$ and $z = e^\tau w(\tau)$. Then the action is [58]

$$S = \int d\tau \frac{\sqrt{(w' + w)^2 - 1}}{w^2} \tag{B.1}$$

The first integral is given by

$$c = \frac{w(w + w') - 1}{w^2\sqrt{(w' + w)^2 - 1}} \tag{B.2}$$

Solving for w' we get

$$w' = -\frac{(w^2 - 1 - c^2w^4) + cw\sqrt{1 - w^2 + c^2w^4}}{w(c^2w^2 - 1)} \tag{B.3}$$

The usual cusp solution is $w = \sqrt{2}$ and $c = 1/2$. We want a solution where $z = \epsilon$ at $\tau = -\infty$. In this case

$$w = \epsilon e^{-\tau} + 1 + \cdots \tag{B.4}$$

should be the behavior as $\tau \to -\infty$. It is possible to see that one can find a solution which obeys these boundary conditions and asymptotes to the usual cusp solution for large τ only for $c = 1/2$. In this case the equation (B.3) simplifies and can be solved as

$$\frac{e^\tau}{\epsilon} = \left(\frac{w+\sqrt{2}}{w-\sqrt{2}}\right)^{\frac{1}{\sqrt{2}}} \frac{1}{1+w} \tag{B.5}$$

We get the solution in an implicit form. We do see that as $\tau \to -\infty$, then $w \to +\infty$ and we recover (B.4) . On the other hand as $\tau \to \infty$, then $w \to \sqrt{2}$. The range of w is $(\sqrt{2}, +\infty)$. w becomes $\sqrt{2}$ when $e^\tau \gg \epsilon$. Thus the solution with boundary conditions at $z = \epsilon$ differs from the solution with boundary condition at $z = 0$ only for e^τ of the order or smaller than ϵ.

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